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The Generalized Boltzmann Equation, Generalized Hydrodynamic Equations and their Applications

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The generalized Boltzmann equation, generalized hydrodynamic equations and their applications

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Contents

The generalization of the Boltzmann equation is realized by taking into account the alteration of the distribution function on scales of the collision time order. The generalized hydrodynamic equations are derived on the basis of the generalized Boltzmann equation. The strict theory of turbulence on the Kolmogorov scale is developed. Examples and issues are given for the shock wave structure and sound wave propagation calculations.

Notation

f_s	s-particle distribution function
$\boldsymbol{v}_i$	velocity of the i-particle
Ω_s	phase volume
α	dimensionless parameter (see (2.4))
λ	scale factor ($\lambda = (nr_b^2)^{-1}$)
V	volume of the system
$\hat{\boldsymbol{r}}_{ib}$	dimensionless vector in the scale of the particle interactions for the i-particle
$\hat{\boldsymbol{r}}_{i\lambda}$	dimensionless vector in the λ-scale
$\mathscr{D}_1\hat{f}_1^1/\mathscr{D}\hat{t}_b$	dimensionless derivative (see (2.14), (2.17))
$\mathrm{d}_1\hat{f}_1^0/\mathrm{d}\hat{t}_{\lambda,L}$	dimensionless derivative (see (2.16))

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$\hat{\boldsymbol{F}}_{ij}$	dimensionless force of interaction between particles i and j
Kn	Knudsen number
$\ln f$	logarithm of the distribution function
$\boldsymbol{B}$	magnetic induction or $B = 1 - \lambda^1/(2\lambda)$ in (4.3)
J	jacobian
$J_{\alpha,\mathrm{el}}$	elastic collision integral for α-component
$J_{\alpha,r}$	non-elastic collision integral for α-component
q_α	charge of the particle for α-component
μ	the number of components in the mixture
m_α	mass of the α-molecule
H	The Boltzmann H-function
$\dfrac{\partial^2 f_\alpha}{\partial \boldsymbol{r}\,\partial \boldsymbol{r}} : \boldsymbol{v}_\alpha \boldsymbol{v}_\alpha$	the typical double vector product
ψ_α^i	collision invariants
ξ	factor in (2.48)
ρ_α	mass density of the α-component
$\mathrm{rot}\,\boldsymbol{B}$	curl of vector B
$\boldsymbol{V}_\alpha$	peculiar velocity of the particles of α-component
$\bar{\varphi}_\alpha$	average value for molecular characteristic
p	static pressure
$\boldsymbol{P}_\alpha$	pressure tensor for α-component
η	dynamical viscosity
ϕ	unknown function in (3.23)
D	tensor (see (3.34)) with components $D_{kl,\alpha}$
Δ	Laplacian
$c_{\alpha 0}^{(\beta,\alpha)}$	Sonine coefficient in (3.35)
Д	tensor with components $Д_{ij}$ in (3.44) and (3.45)
$\boldsymbol{\Phi}_{i,w}$	boundary value in (3.51)
$\delta_{\alpha\beta}$	delta function in (3.53)
$\hat{k}$	dimensionless wave number in (4.4)
κ	γ^{-1}
A	see (4.3), where $A = \eta^1/\eta$
c_0	sound speed in the Euler approximation

Signs

ˆ	dimensionless value
·	scalar product
:	double tensor product
—	average molecular values
×	vector product

1. Introduction

In recent years considerable efforts have been done in the derivation of generalized hydrodynamic equations (GHE) using the generalized Boltzmann equation (GBE) (Alexeev 1987, 1988, 1990*a*–*d*, 1992). It is well known, that the Boltzmann equation is valid for the hydrodynamic scales and also for scales related to the mean free path between collisions. The proposed generalization of the classical Boltzmann kinetic theory takes into account the alteration of the distribution function on scales of the collision time order. The fact of great importance is that this additional term is of Knudsen number order. The GBE leads to a new system of hydrodynamic equations: GHE, and the classical Euler and Navier–Stokes equations are only a particular case of this new theory. As a result we can build a strict theory of turbulence on Kolmogorov's scale of turbulence. The applications of the GHE and the GBE are considered including the calculation of forced sound-wave speed and damping of the sound for arbitrary relation between the frequencies of forced oscillations and molecular collisions. Also with the help of the GHE the structure of shock waves for arbitrary Mach numbers is calculated.

2. The generalized Boltzmann equation

(*a*) *Application of Bogolubov chain of equations in the GBE theory*

Let us consider the chain of Bogolubov kinetic equations. In usual nomenclature we can write the next equation for the s-particle distribution function f_s

$$\frac{\partial f_s}{\partial t}+\sum_{i=1}^{s} \boldsymbol{v}_i\cdot\frac{\partial f_s}{\partial \boldsymbol{r}_i}+\sum_{i,j=1}^{s} \boldsymbol{F}_{ij}\cdot\frac{\partial f_s}{\partial \boldsymbol{p}_i}+\sum_{i=1}^{s} \boldsymbol{F}_i\cdot\frac{\partial f_s}{\partial \boldsymbol{p}_i}=-\sum_{i=1}^{s}(\mathcal{N}-s)\frac{\partial}{\partial \boldsymbol{p}_i}\cdot\int \boldsymbol{F}_{i,s+1}f_{s+1}\,\mathrm{d}\Omega_{s+1}, \quad (2.1)$$

where Ω_{s+1} is phase volume, $\mathrm{d}\Omega_{s+1}=\mathrm{d}\boldsymbol{r}_{s+1}\,\mathrm{d}\boldsymbol{p}_{s+1}$, and N the number of particles in the system. We are taking into account internal forces $\boldsymbol{F}_{ij}$ acting between particles of the same chemical structure and the external forces $\boldsymbol{F}_i$. The Bogolubov equation (2.1) can be written in the dimensionless form. For this purpose we introduce three groups of scales connected with the particle collisions (the scales for the velocity v_0, the collision time τ_b and the interaction length r_b), the particle evolution between collisions (the scales $v_{0\lambda}$, τ_λ and the mean free path between collision $\lambda=(nr_b^2)^{-1}$, where n is the number density), the hydrodynamic scales (v_L, τ_L, L). We call these scales b, λ and L scales respectively. The dimensionless values we denote as $\hat{}$, for example we have

$$t=r_b\hat{t}v_0^{-1},\quad F_{ij}=F_0\hat{F}_{ij},\quad f_s=\hat{f}_s p_0^{-3s}V^{-s}, \quad (2.2)$$

where p_0 is the impulse scale.

The Bogolubov equation is transformed into

$$\frac{\partial \hat{f}_s}{\partial \hat{t}}+\sum_{i=1}^{s} \hat{\boldsymbol{v}}_i\cdot\frac{\partial \hat{f}_s}{\partial \hat{\boldsymbol{r}}_i}+\sum_{i,j=1}^{s} \hat{\boldsymbol{F}}_{ij}\cdot\frac{\partial \hat{f}_s}{\partial \hat{\boldsymbol{p}}_i}+\alpha\sum_{i=1}^{s} \hat{\boldsymbol{F}}_i\cdot\frac{\partial \hat{f}_s}{\partial \hat{\boldsymbol{p}}_i}=-\beta\sum_{i=1}^{s}(\mathcal{N}-s)\frac{\partial}{\partial \hat{\boldsymbol{p}}_i}\cdot\int \hat{\boldsymbol{F}}_{i,s+1}\hat{f}_{s+1}\,\mathrm{d}\hat{\Omega}_{s+1}, \quad (2.3)$$

where we introduce the parameters

$$\alpha=F_0^{-1}F_{0\lambda},\quad \beta=r_b^3V^{-1}, \quad (2.4)$$

where V is the volume of the system. Using the method of many scales (Kogan 1969; Naife 1976; Alexeev 1982) we introduce additional space, impulse and time variables for the s-particle distribution function f_s:

$$\hat{f}_s=\hat{f}_s(\hat{t}_b,\hat{\boldsymbol{r}}_{ib},\hat{\boldsymbol{p}}_i;t_\lambda,\hat{\boldsymbol{r}}_{i\lambda},\hat{\boldsymbol{p}}_{i\lambda};\hat{t}_L,\hat{\boldsymbol{r}}_{iL},\hat{\boldsymbol{p}}_{iL};\epsilon), \quad (2.5)$$

where $\epsilon=nr_b^3$ is a small parameter. Using the condition $s\ll N$ we get the next dimensionless kinetic equation:

$$\frac{\partial \hat{f}_s}{\partial \hat{t}}+\sum_{i=1}^{s} \hat{\boldsymbol{v}}_i\cdot\frac{\partial \hat{f}_s}{\partial \hat{\boldsymbol{r}}_i}+\sum_{i,j=1}^{s} \hat{\boldsymbol{F}}_{ij}\cdot\frac{\partial \hat{f}_s}{\partial \hat{\boldsymbol{p}}_i}+\alpha\sum_{i=1}^{s} \hat{\boldsymbol{F}}_i\cdot\frac{\partial \hat{f}_s}{\partial \hat{\boldsymbol{p}}_i}=-\epsilon\sum_{i=1}^{s}\frac{\partial}{\partial \hat{\boldsymbol{p}}_i}\cdot\int \hat{\boldsymbol{F}}_{i,s+1}\hat{f}_{s+1}\,\mathrm{d}\hat{\Omega}_{s+1}. \quad (2.6)$$

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In the considered case we do not need any additional variables for velocity; for other dimensionless variables we use scale factors

$$\tau_\lambda = \lambda v_{0\lambda}^{-1}, \quad r_b = \tau_b v_0, \quad \epsilon_2 = v_{0\lambda} v_0^{-1}, \quad \epsilon_3 = v_{0L} v_{0\lambda}^{-1}. \tag{2.7}$$

As a result we have

$$\hat{r}_{ib} = r_{ib}/r_b, \quad \hat{t}_b = t/t_b, \quad \hat{r}_{i\lambda} = r_i/\lambda = \epsilon \hat{r}_{ib}. \tag{2.8}$$

By using analogical transformation

$$\hat{t}_\lambda = \frac{t}{\tau_\lambda} = \epsilon \hat{t}_b, \quad \hat{r}_{iL} = \frac{r_b}{L}\hat{r}_{ib} = \frac{r_b}{\lambda}\frac{\lambda}{L}\hat{r}_{ib} = \epsilon\epsilon_1 \hat{r}_{ib}, \tag{2.9}$$

and at last for hydrodynamical time we find

$$\hat{t}_L = \frac{\tau_b}{\tau_L}\hat{t}_b = \frac{\tau_b}{\tau_\lambda}\frac{\tau_\lambda}{\tau_L}\hat{t}_b = \epsilon\epsilon_1 \hat{t}_b, \tag{2.10}$$

where $\epsilon_1 = \lambda/L = Kn$ (the Knudsen number); any restrictions for Knudsen numbers are not considered. Let us expand now f_s in a power series in ϵ:

$$\hat{f}_s = \sum_{\nu=0}^{\infty} \hat{f}_s^\nu(\hat{t}_b, \hat{\boldsymbol{r}}_{ib}, \hat{\boldsymbol{p}}_i, \hat{t}_\lambda, \hat{\boldsymbol{r}}_{i\lambda}, \hat{\boldsymbol{p}}_{i\lambda}, \hat{t}_L, \hat{\boldsymbol{r}}_{iL}, \hat{\boldsymbol{p}}_{iL})\, \epsilon^\nu. \tag{2.11}$$

Let us find the derivatives using (2.8)–(2.10)

$$\frac{\partial \hat{f}_s}{\partial \hat{t}_b} = \frac{\partial \hat{f}_s^0}{\partial \hat{t}_b} + \epsilon \frac{\partial \hat{f}_s^1}{\partial \hat{t}_b} + \frac{\partial \hat{f}_s^0}{\partial \hat{t}_\lambda}\frac{\partial \hat{t}_\lambda}{\partial \hat{t}_b} + \frac{\partial \hat{f}_s^0}{\partial \hat{t}_L}\frac{\partial \hat{t}_L}{\partial \hat{t}_b} + \ldots = \frac{\partial \hat{f}_s^0}{\partial \hat{t}_b} + \epsilon\left(\frac{\partial \hat{f}_s^1}{\partial \hat{t}_b} + \epsilon_2 \frac{\partial \hat{f}_s^0}{\partial \hat{t}_\lambda} + \epsilon_1\epsilon_2\epsilon_3 \frac{\partial \hat{f}_s^0}{\partial \hat{t}_L}\right), \tag{2.12}$$

$$\frac{\partial \hat{f}_s}{\partial \hat{\boldsymbol{r}}_{ib}} = \frac{\partial \hat{f}_s^0}{\partial \hat{\boldsymbol{r}}_{ib}} + \epsilon\left(\frac{\partial \hat{f}_s^1}{\partial \hat{\boldsymbol{r}}_{ib}} + \frac{\partial \hat{f}_s^0}{\partial \hat{\boldsymbol{r}}_{i\lambda}} + \epsilon_1 \frac{\partial \hat{f}_s^0}{\partial \hat{\boldsymbol{r}}_{iL}}\right). \tag{2.13}$$

Substituting (2.12) and (2.13) in (2.6) and equalizing the coefficients by the equal power of ϵ, we find

$$\frac{\mathscr{D}_s \hat{f}_s^0}{\mathscr{D}\hat{t}_b} \equiv \frac{\partial \hat{f}_s^0}{\partial \hat{t}_b} + \sum_{i=1}^{s} \hat{\boldsymbol{v}}_i \cdot \frac{\partial \hat{f}_s^0}{\partial \hat{\boldsymbol{r}}_{ib}} + \sum_{i,j=1}^{s} \hat{\boldsymbol{F}}_{ij} \cdot \frac{\partial \hat{f}_s^0}{\partial \hat{\boldsymbol{p}}_i} + \alpha \sum_{i=1}^{s} \hat{\boldsymbol{F}}_i \cdot \frac{\partial \hat{f}_s^0}{\partial \boldsymbol{p}_i} = 0, \tag{2.14}$$

$$\frac{\mathscr{D}_s \hat{f}_s^1}{\mathscr{D}\hat{t}_b} + \frac{\mathrm{d}_s \hat{f}_s^0}{\mathrm{d}\hat{t}_{\lambda,L}} = -\sum_{i=1}^{s} \frac{\partial}{\partial \hat{\boldsymbol{p}}_i} \cdot \int \hat{\boldsymbol{F}}_{i,s+1} \hat{f}_{s+1}^0 \, \mathrm{d}\hat{\Omega}_{s+1}, \tag{2.15}$$

where

$$\frac{\mathrm{d}_s}{\mathrm{d}\hat{t}_{\lambda,L}} = \epsilon_2 \frac{\partial}{\partial \hat{t}_\lambda} + \sum_{i=1}^{s} \hat{\boldsymbol{v}}_i \cdot \frac{\partial}{\partial \hat{\boldsymbol{r}}_{i\lambda}} + \epsilon_1\left(\epsilon_2\epsilon_3 \frac{\partial}{\partial \hat{t}_L} + \sum_{i=1}^{s} \hat{\boldsymbol{v}}_i \cdot \frac{\partial}{\partial \hat{\boldsymbol{r}}_{iL}}\right) + \epsilon_2 \sum_{i=1}^{s} \hat{\boldsymbol{F}}_i \cdot \frac{\partial}{\partial \hat{\boldsymbol{p}}_{i\lambda}}$$
$$+ \frac{F_0}{F_{0\lambda}}\epsilon_2 \sum_{ij=1}^{s} \hat{\boldsymbol{F}}_{ij} \cdot \frac{\partial}{\partial \hat{\boldsymbol{p}}_{i\lambda}} + \frac{\epsilon_2}{\epsilon_3}\sum_{i=1}^{s} \hat{\boldsymbol{F}}_i \cdot \frac{\partial}{\partial \hat{\boldsymbol{p}}_{iL}} + \frac{F_0 \epsilon_2}{F_0 \epsilon_3}\sum_{i,j=1}^{s} \hat{\boldsymbol{F}}_{ij} \cdot \frac{\partial}{\partial \hat{\boldsymbol{p}}_{iL}}. \tag{2.16}$$

The kinetic equation for the one-particle distribution function can be written

$$\frac{\mathscr{D}_1 \hat{f}_1^1}{\mathscr{D}\hat{t}_b} + \frac{\mathrm{d}_1 \hat{f}_1^0}{\mathrm{d}\hat{t}_{\lambda,L}} = -\frac{\partial}{\partial \hat{\boldsymbol{p}}_1} \cdot \int \hat{\boldsymbol{F}}_{12} \hat{f}_2^0 \, \mathrm{d}\hat{\boldsymbol{r}}_{2b} \, \mathrm{d}\hat{\boldsymbol{p}}_2. \tag{2.17}$$

Introducing the new variable $\boldsymbol{x}_{12} = \boldsymbol{r}_{1b} - \boldsymbol{r}_{2b}$ and observing that

$$\frac{\mathscr{D}_2 \hat{f}_2^0}{\mathscr{D}\hat{t}_b} \equiv \frac{\partial \hat{f}_2^0}{\partial \hat{t}_b} + \boldsymbol{v}_1 \cdot \frac{\partial \hat{f}_2^0}{\partial \hat{\boldsymbol{r}}_{1b}} + (\hat{\boldsymbol{v}}_1 - \hat{\boldsymbol{v}}_2) \cdot \frac{\partial \hat{f}_2^0}{\partial \hat{\boldsymbol{x}}_{12,b}} + \sum_{i,j=1}^{2} \hat{\boldsymbol{F}}_{ij} \cdot \frac{\partial \hat{f}_2^0}{\partial \hat{\boldsymbol{p}}_i} + \alpha \sum_{i=1}^{2} \hat{\boldsymbol{F}}_i \cdot \frac{\partial \hat{f}_2^0}{\partial \hat{\boldsymbol{p}}_i} = 0, \tag{2.18}$$

we therefore get

$$-\boldsymbol{F}_{12}\cdot\frac{\partial \hat{f}_2^0}{\partial \hat{\boldsymbol{p}}_1}=\frac{\partial \hat{f}_2^0}{\partial \hat{t}_b}+\hat{\boldsymbol{v}}_1\cdot\frac{\partial \hat{f}_2^0}{\partial \hat{\boldsymbol{r}}_{1b}}+(\hat{\boldsymbol{v}}_1-\hat{\boldsymbol{v}}_2)\cdot\frac{\partial \hat{f}_2^0}{\partial \hat{\boldsymbol{x}}_{12,b}}+\hat{\boldsymbol{F}}_{21}\cdot\frac{\partial \hat{f}_2^0}{\partial \hat{\boldsymbol{p}}_2}+\alpha\sum_{i=1}^{2}\hat{\boldsymbol{F}}_i\cdot\frac{\partial \hat{f}_2^0}{\partial \hat{\boldsymbol{p}}_i}. \tag{2.19}$$

Let us transform the integral in the right-hand part of (2.15), supposing that the force of interaction between the first and the second particles does not depend on the velocity:

$$-\int \hat{\boldsymbol{F}}_{12}\cdot\frac{\partial \hat{f}_2^0}{\partial \hat{\boldsymbol{p}}_1}\,\mathrm{d}\hat{\boldsymbol{r}}_{2b}\,\mathrm{d}\hat{\boldsymbol{p}}_2=\int(\hat{\boldsymbol{v}}_1-\hat{\boldsymbol{v}}_2)\cdot\frac{\partial \hat{f}_2^0}{\partial \hat{\boldsymbol{x}}_{12},b}\,\mathrm{d}\hat{\boldsymbol{r}}_{2,b}\,\mathrm{d}\hat{\boldsymbol{p}}_2$$
$$+\int\left(\frac{\partial \hat{f}_2^0}{\partial \hat{t}_b}+\hat{\boldsymbol{v}}_1\cdot\frac{\partial \hat{f}_2^0}{\partial \hat{\boldsymbol{r}}_{1b}}+\alpha\sum_{i=1}^{2}\hat{\boldsymbol{F}}_i\cdot\frac{\partial \hat{f}_2^0}{\partial \hat{\boldsymbol{p}}_i}\right)\mathrm{d}\hat{\boldsymbol{r}}_{2,b}\,\mathrm{d}\hat{\boldsymbol{p}}_2+\int \hat{\boldsymbol{F}}_{21}\cdot\frac{\partial \hat{f}_2^0}{\partial \hat{\boldsymbol{p}}_2}\,\mathrm{d}\hat{\boldsymbol{r}}_{2,b}\,\mathrm{d}\hat{\boldsymbol{p}}_2. \tag{2.20}$$

The last integral in the right-hand part of (2.20) can be transformed into the surface integral in the velocity space. Therefore this integral is equal to zero, because the distribution function tends to zero if the velocity tends to infinity. Hence we have

$$\frac{\mathscr{D}_1\hat{f}_1^1}{\mathscr{D}\hat{t}_b}+\frac{d_1\hat{f}_1^0}{\mathrm{d}\hat{t}_{\lambda,L}}=\int(\hat{\boldsymbol{v}}_1-\hat{\boldsymbol{v}}_2)\cdot\frac{\partial \hat{f}_2^0}{\partial \hat{\boldsymbol{x}}_{12,b}}\,\mathrm{d}\hat{\boldsymbol{r}}_{2b}\,\mathrm{d}\hat{\boldsymbol{p}}_2+\int\left(\frac{\partial \hat{f}_2^0}{\partial \hat{t}_b}+\hat{\boldsymbol{v}}_1\cdot\frac{\partial \hat{f}_2^0}{\partial \hat{\boldsymbol{r}}_{1b}}+\alpha\sum_{i=1}^{2}\hat{\boldsymbol{F}}_i\cdot\frac{\partial \hat{f}_2^0}{\partial \hat{\boldsymbol{p}}_i}\right)\mathrm{d}\hat{\boldsymbol{r}}_{2b}\,\mathrm{d}\hat{\boldsymbol{p}}_2. \tag{2.21}$$

The equation (2.21) must be used for the calculation of $\hat{f}_1^0$, but this equation contains the term, belonging to the senior equations in the Bogolubov chain of equations, in the right-hand side (because of $\hat{f}_2^0$) and the term, connected with $\hat{f}_1^1$, belonging to the senior term of the power series (2.11). It is well known that the two-particle distribution function $\hat{f}_2^0$ (which equals $\hat{f}_{2,\epsilon=0}$) can be performed in rarefied gas as

$$\hat{f}_2^0=\hat{f}_{1,1}^0\hat{f}_{1,2}^0. \tag{2.22}$$

Therefore the last integral term in (2.21) can be written, using the condition of normalization

$$\int f_1\,\mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{p}=1, \tag{2.23}$$

in the form

$$\int\left(\frac{\partial \hat{f}_2^0}{\partial \hat{t}_b}+\hat{\boldsymbol{v}}_1\cdot\frac{\partial \hat{f}_2^0}{\partial \hat{\boldsymbol{r}}_{1b}}+\alpha\hat{\boldsymbol{F}}_1\cdot\frac{\partial \hat{f}_2^0}{\partial \hat{\boldsymbol{p}}_1}+\alpha\hat{\boldsymbol{F}}_2\cdot\frac{\partial \hat{f}_2^0}{\partial \hat{\boldsymbol{p}}_2}\right)\mathrm{d}\hat{\boldsymbol{r}}_{2b}\,\mathrm{d}\hat{\boldsymbol{p}}_2=\frac{\partial \hat{f}_{1,1}^0}{\partial \hat{t}_b}+\hat{\boldsymbol{v}}_1\cdot\frac{\partial \hat{f}_{1,1}^0}{\partial \hat{\boldsymbol{r}}_{1b}}+\alpha\hat{\boldsymbol{F}}_1\cdot\frac{\partial \hat{f}_{1,1}^0}{\partial \hat{\boldsymbol{p}}_{1b}}. \tag{2.24}$$

But these terms equal zero because of (2.14), if we take into account that $\boldsymbol{F}_{11}\equiv 0$. Kinetic equation (2.21) can be simplified as

$$\frac{\mathscr{D}_1\hat{f}_1^1}{\mathscr{D}\hat{t}_b}+\frac{d_1\hat{f}_1^0}{\mathrm{d}\hat{t}_{\lambda,L}}=\int(\hat{\boldsymbol{v}}_2-\hat{\boldsymbol{v}}_1)\cdot\frac{\partial}{\partial \hat{\boldsymbol{x}}_{12,b}}(\hat{f}_{1,1}^0 f_{1,2}^0)\,\mathrm{d}\hat{\boldsymbol{x}}_{12,b}\,\mathrm{d}\hat{\boldsymbol{p}}_2. \tag{2.25}$$

The integral term in the right-hand side of (2.25) (after integration in r_b-scale) leads to the usual Boltzmann collision term. Nevertheless the left-hand side of (2.25) contains the derivative mentioned above. I should emphasize that this term, generally speaking, is of the same order as the second term in the left-hand side of (2.25). The usual argument for omitting $\mathscr{D}_1\hat{f}_1^1/\mathscr{D}\hat{t}_b$ leads to the affirmation that this term is apparently small because it takes into account the alteration of the distribution function in the t_b scale. But this term must be considered more thoroughly, because, roughly speaking, it is proportional to ratio of the small value $\hat{f}_1^1$ to the small value of $\hat{t}_b$, which leads to the appearance of the term of the same order as $d_1\hat{f}_1^0/\mathrm{d}\hat{t}_{\lambda,L}$.

Therefore, we must discuss the approximation of this derivative in the t_b scale (Alexeev 1988). In a definite sense we have a problem of the same character, as by approximation of $\hat{f}_2^0$ (see 2.22) it is obvious that

$$\frac{\mathscr{D}_1 \hat{f}_1^1}{\mathscr{D}\hat{t}_b} = \frac{\mathscr{D}_1}{\mathscr{D}\hat{t}_b}\left[\left(\frac{\partial \hat{f}_1}{\partial \epsilon}\right)_{\epsilon=0}\right]. \tag{2.26}$$

In the t_b scale it is natural to consider dynamical variables $\hat{t}_b, \hat{\boldsymbol{r}}_b, \epsilon$ as correlated values. Therefore, it is possible to write

$$\frac{\mathscr{D}_1}{\mathscr{D}\hat{t}_b}\left[\left(\frac{\partial \hat{f}_1}{\partial \epsilon}\right)_{\epsilon=0}\right] = \frac{\mathscr{D}_1}{\mathscr{D}\hat{t}_b}\left[\frac{\partial \hat{f}_1}{\partial \hat{t}_b}\left(\frac{\partial \hat{t}_b}{\partial \epsilon}\right)_{\epsilon=0} + \frac{\partial \hat{f}_1}{\partial \hat{\boldsymbol{r}}_b}\cdot\frac{\partial \hat{\boldsymbol{r}}_b}{\partial \hat{t}_b}\left(\frac{\partial \hat{t}_b}{\partial \epsilon}\right)_{\epsilon=0} + \frac{\partial \hat{f}_1}{\partial \hat{\boldsymbol{v}}}\cdot\frac{\partial \hat{\boldsymbol{v}}}{\partial \hat{t}_b}\left(\frac{\partial \hat{t}_b}{\partial \epsilon}\right)_{\epsilon=0}\right]. \tag{2.27}$$

We introduce the next approximation of the derivative (2.26);

$$\frac{\mathscr{D}_1 \hat{f}_1^1}{\mathscr{D}\hat{t}_b} = \pm\frac{\mathscr{D}}{\mathscr{D}\hat{t}_b}\left[\left|\frac{\partial \hat{t}_b}{\partial \epsilon}\right|_{\epsilon=0}\frac{\mathscr{D}\hat{f}_1^0}{\mathscr{D}\hat{t}_b}\right]. \tag{2.28}$$

The parameter $|\partial \hat{t}_b/\partial \epsilon|_{\epsilon=0}$ can be discussed, using the terminology of time relaxation. In fact, let us return to the dimensional form of the kinetic equation. The one-particle distribution function (corresponding to the first term of expansion in the power series (2.14)) we denote, as usual, f.

The generalized Boltzmann equation takes the form

$$\frac{\mathscr{D}f}{\mathscr{D}t} \pm \frac{\mathscr{D}}{\mathscr{D}t}\left[\tau\frac{\mathscr{D}f}{\mathscr{D}t}\right] = J_{\mathrm{B}}, \tag{2.29}$$

where J_{B} is Boltzmann collision integral and

$$\tau = \epsilon^0/(\partial\epsilon/\partial t)_{\epsilon=0}, \quad \epsilon^0 = nr_b^3. \tag{2.30}$$

The problem of the choice of sign before τ we discuss later in this section, but now we must define the physical sense of the parameter τ. From the physical point of view the derivative $(\partial\epsilon/\partial t)_{\epsilon=0}$ corresponds to the number of particles entering in the interaction volume per unit of time if in the initial time moment there are no particles in this volume. Obviously, the number of these particles equals the collision number occurring in the volume r_b^3. Therefore the parameter τ is the mean time between collisions. For example, for the model of hard spheres, in the continual limit, τ can be connected with dynamical viscosity (Chapman & Cowling 1939) η:

$$\tau p = 0.786\eta. \tag{2.31}$$

The approximation (2.28) is realized on a time interval of collision time order. Therefore there is no problem of secular terms.

By investigation of the hydrodynamical problem it is useful to transform (2.29) to the dimensionless form, using the following: hydrodynamical time τ_L; corresponding length, L; mean molecular velocity, and also the value of the distribution function, defined as the ratio of the character number of molecules in unit volume to the mean thermal velocity cubed.

Then the generalized Boltzmann equation can be written in the following elegant form (more simple approximation in the hydrodynamic limit):

$$\frac{\mathscr{D}\hat{f}}{\mathscr{D}\hat{t}} \pm Kn\frac{\mathscr{D}}{\mathscr{D}\hat{t}}\frac{\mathscr{D}\hat{f}}{\mathscr{D}\hat{t}} = \frac{1}{Kn}\hat{J}_B. \tag{2.32}$$

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As a conclusion, it is possible to state that making allowance for alteration of the distribution function in the τ^b scale, we receive the additional term on the left-hand side of the Boltzmann equation, which is of the Knudsen number order. The collision integral $\hat{J}_B$ must be written in the classical form, because it contains only the distribution function of zero order.

Let us discuss now the problem of the choice of sign before τ. This problem is connected with the H-theorem of Boltzmann and with the Prigogine principle of the entropy rate (Nicolis & Prigogine 1977):

$$\mathrm{d}H/\mathrm{d}t \leqslant 0 \quad \text{(Boltzmann principle)}, \tag{2.33}$$

$$\mathrm{d}^2H/\mathrm{d}t^2 \geqslant 0 \quad \text{(Prigogine principle)}, \tag{2.34}$$

where
$$H = \int f \ln f \,\mathrm{d}\hat{\boldsymbol{v}}.$$

I intend to show that the existence of the Boltzmann and Prigogine principles leads to a negative sign in equation (2.32). Following the usual procedure we consider a homogeneous gas without any external forces. Therefore the GBE has the form

$$\frac{\partial f}{\partial t} \pm \tau \frac{\partial^2 f}{\partial t^2} = J_{\mathrm{B}}. \tag{2.35}$$

After multiplication of both sides of (2.35) by $\ln f$ and simple transformations we get

$$\frac{\partial}{\partial t}(f \ln f) \pm \tau \frac{\partial^2}{\partial t^2}(f \ln f) = \pm \tau \frac{1}{f}\left(\frac{\partial f}{\partial t}\right)^2 + (1 + \ln f) J_{\mathrm{B}}. \tag{2.36}$$

Integration over velocity $\boldsymbol{v}$ leads to the result

$$\frac{\partial H}{\partial t} \pm \tau \frac{\partial^2 H}{\partial t^2} = \pm \tau \int \frac{1}{f}\left(\frac{\partial f}{\partial t}\right)^2 \mathrm{d}\boldsymbol{v} + \int (1 + \ln f) J_{\mathrm{B}} \,\mathrm{d}\boldsymbol{v}. \tag{2.37}$$

The last integral on the left-hand side of (2.37) can be transformed with the help of the principle of microscopical reversibility

$$\int (1 + \ln f) J_{\mathrm{B}} \,\mathrm{d}\boldsymbol{v} = -\frac{1}{4}\int \ln \frac{f' f_1'}{f f_1} (f_1' f' - f f_1)\, g b \,\mathrm{d}b \,\mathrm{d}\varphi \,\mathrm{d}\boldsymbol{v}_1 \,\mathrm{d}\boldsymbol{v}, \tag{2.38}$$

where g is the relative velocity of two colliding particles, b is the impact parameter and φ is the azimuthal angle. Therefore we have

$$\frac{\partial H}{\partial t} \pm \tau \frac{\partial^2 H}{\partial t^2} = \pm \tau \int \frac{1}{f}\left(\frac{\partial f}{\partial t}\right)^2 \mathrm{d}\boldsymbol{v} - \frac{1}{4}\int \ln \frac{f' f_1'}{f f_1} (f_1' f' - f_1 f)\, g b \,\mathrm{d}b \,\mathrm{d}\varphi \,\mathrm{d}\boldsymbol{v}_1 \,\mathrm{d}\boldsymbol{v}. \tag{2.39}$$

Now we can state using the Prigogine and Boltzmann principles that the inequalities

$$\frac{\partial H}{\partial t} - \tau \frac{\partial^2 H}{\partial t^2} \leqslant 0, \quad \frac{\partial H^a}{\partial t} \leqslant 0, \quad H^a = H - \tau \frac{\partial H}{\partial t} \tag{2.40}$$

are valid always if we choose minus before parameter τ. Consequently we get the kinetic equation

$$\frac{\mathscr{D} f}{\mathscr{D} t} - \frac{\mathscr{D}}{\mathscr{D} t}\left(\tau \frac{\mathscr{D} f}{\mathscr{D} t}\right) = J_{\mathrm{B}}. \tag{2.41}$$

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For the multicomponent system we have obviously

$$\frac{\mathscr{D}f_\alpha}{\mathscr{D}t}-\frac{\mathscr{D}}{\mathscr{D}t}\left(\tau\,\frac{\mathscr{D}f_\alpha}{\mathscr{D}t}\right)=J_{\mathrm{B},\alpha}$$

or

$$\frac{\mathscr{D}f_\alpha}{\mathscr{D}t}-\frac{\mathscr{D}\tau}{\mathscr{D}t}\frac{\mathscr{D}f_\alpha}{\mathscr{D}t}-\tau\frac{\mathscr{D}}{\mathscr{D}t}\frac{\mathscr{D}f_\alpha}{\mathscr{D}t}=J_{\mathrm{B},\alpha},$$

where

$$\frac{\mathscr{D}}{\mathscr{D}t}=\frac{\partial}{\partial t}+\boldsymbol{v}_\alpha\cdot\frac{\partial}{\partial \boldsymbol{r}}+\boldsymbol{F}_\alpha\cdot\frac{\partial}{\partial \boldsymbol{v}_\alpha},$$

$$\frac{\mathscr{D}}{\mathscr{D}t}\frac{\mathscr{D}f_\alpha}{\mathscr{D}t}=\frac{\partial^2 f_\alpha}{\partial t^2}+2\boldsymbol{v}_\alpha\cdot\frac{\partial^2 f_\alpha}{\partial \boldsymbol{r}\,\partial t}+2\boldsymbol{F}_\alpha\cdot\frac{\partial^2 f_\alpha}{\partial \boldsymbol{v}_\alpha \partial t}+\frac{\partial^2 f_\alpha}{\partial \boldsymbol{r}\,\partial \boldsymbol{r}}:\boldsymbol{v}_\alpha\boldsymbol{v}_\alpha+\boldsymbol{F}_\alpha\cdot\frac{\partial f_\alpha}{\partial \boldsymbol{r}}+2\frac{\partial^2 f_\alpha}{\partial \boldsymbol{r}\,\partial \boldsymbol{v}_\alpha}:\boldsymbol{F}_\alpha\boldsymbol{v}_\alpha$$
$$+\frac{\partial \boldsymbol{F}_\alpha}{\partial t}\cdot\frac{\partial f_\alpha}{\partial \boldsymbol{v}_\alpha}+\frac{\partial f_\alpha}{\partial \boldsymbol{v}_\alpha}\boldsymbol{v}_\alpha:\frac{\partial}{\partial \boldsymbol{r}}\boldsymbol{F}_\alpha+\frac{\partial^2 f_\alpha}{\partial \boldsymbol{v}_\alpha\partial \boldsymbol{v}_\alpha}:\boldsymbol{F}_\alpha\boldsymbol{F}_\alpha+\frac{\partial f_\alpha}{\partial \boldsymbol{v}_\alpha}\boldsymbol{F}_\alpha:\frac{\partial}{\partial \boldsymbol{v}_\alpha}\boldsymbol{F}_\alpha.$$

Because of the fundamental significance of this result we now discuss the GBE (2.41) from another point of view using the 'physical' method of derivation of Boltzmann equation.

(b) *The physical method of derivation of the* GBE

Now we can go on to derive the GBE from a phenomenological or 'physical' point of view.

The alteration of the number of α-molecules must be connected with encounters and defined with the help of the Boltzmann collision integral:

$$f_\alpha\left[\boldsymbol{r}+\boldsymbol{v}_\alpha\,\mathrm{d}t+\tfrac{1}{2}\boldsymbol{F}_\alpha(\mathrm{d}t)^2,\boldsymbol{v}_\alpha+\boldsymbol{F}_\alpha\,\mathrm{d}t+\frac{1}{2}\frac{\partial \boldsymbol{F}_\alpha}{\partial t}(\mathrm{d}t)^2,t+\mathrm{d}t\right]J-f_\alpha(\boldsymbol{r},\boldsymbol{v}_\alpha,t)=[J_{\alpha,\mathrm{el}}+J_{\alpha,r}]\,\mathrm{d}t. \tag{2.42}$$

We intend to take into account the terms, which are of $O((\mathrm{d}t)^2)$. The calculation of the jacobian J with this accuracy leads to the result

$$J=1+\left(\frac{q_\alpha}{m_\alpha}\right)^2 B^2(\mathrm{d}t)^2-\frac{1}{2}\frac{\partial}{\partial \boldsymbol{r}}\cdot\boldsymbol{F}_\alpha(\mathrm{d}t)^2, \tag{2.43}$$

where B is magnetic induction, q_α is the charge of α-species, $\boldsymbol{F}_\alpha$ is an external force acting on unit mass of species α. Expanding now the left-hand side of equation (2.42) by Taylor's theorem and keeping the $O(\mathrm{d}t)^2$-terms we find, omitting for simplicity the terms connected with the external forces,

$$\frac{\partial f_\alpha}{\partial t}+\boldsymbol{v}_\alpha\cdot\frac{\partial f_\alpha}{\partial \boldsymbol{r}}\pm\tau\left\{\frac{\partial^2 f_\alpha}{\partial t^2}+\frac{\partial^2 f_\alpha}{\partial \boldsymbol{r}\,\partial \boldsymbol{r}}:\boldsymbol{v}_\alpha\boldsymbol{v}_\alpha+2\frac{\partial^2 f_\alpha}{\partial \boldsymbol{r}\,\partial t}\cdot\boldsymbol{v}_\alpha\right\}=J_{\alpha,\mathrm{el}}+J_{\alpha,r}, \tag{2.44}$$

where $\tau\equiv \mathrm{d}t/2$, and the sign $:$ denotes the double tensor product. There is a double sign before parameter τ because of the possibility of another approximation of the distribution function:

$$f_\alpha(\boldsymbol{r},\boldsymbol{v}_\alpha,t)\frac{\mathrm{d}[\boldsymbol{r}^t,\boldsymbol{v}_\alpha^t]}{\mathrm{d}[\boldsymbol{r}^{t-\mathrm{d}t},\boldsymbol{v}_\alpha^{t-\mathrm{d}t}]}-f_\alpha(\boldsymbol{r}-\Delta\boldsymbol{r},\boldsymbol{v}_\alpha-\Delta\boldsymbol{v}_\alpha,t-\mathrm{d}t)=[J_{\alpha,\mathrm{el}}+J_{\alpha,r}]\,\mathrm{d}t. \tag{2.45}$$

Equation (2.44), from the phenomenological point of view, is obviously of the same form as (2.32), received from the BBGKI-hierarchy of equations. Of course, we must take into account the more general character of conditions used by derivation (2.32). Therefore we must choose a minus sign before τ in (2.44). Let us write the general Boltzmann equation once more:

$$\frac{\partial f_\alpha}{\partial t}+\boldsymbol{v}_\alpha\cdot\frac{\partial f_\alpha}{\partial \boldsymbol{r}}+\boldsymbol{F}_\alpha\cdot\frac{\partial f_\alpha}{\partial \boldsymbol{v}_\alpha}-\tau\left\{\boldsymbol{F}_\alpha\cdot\frac{\partial f_\alpha}{\partial \boldsymbol{r}}+\frac{\partial \boldsymbol{F}_\alpha}{\partial t}\cdot\frac{\partial f_\alpha}{\partial \boldsymbol{v}_\alpha}+\frac{\partial^2 f_\alpha}{\partial t^2}+\frac{\partial^2 f_\alpha}{\partial \boldsymbol{r}\,\partial \boldsymbol{r}}:\boldsymbol{v}_\alpha\boldsymbol{v}_\alpha+\frac{\partial f_\alpha}{\partial \boldsymbol{v}_\alpha}\boldsymbol{v}_\alpha:\frac{\partial}{\partial \boldsymbol{r}}\boldsymbol{F}_\alpha\right.$$
$$\left.+\frac{\partial f_\alpha}{\partial \boldsymbol{v}_\alpha}\boldsymbol{F}_\alpha:\frac{\partial}{\partial \boldsymbol{v}_\alpha}\boldsymbol{F}_\alpha+\frac{\partial^2 f_\alpha}{\partial \boldsymbol{v}_\alpha\,\partial \boldsymbol{v}_\alpha}:\boldsymbol{F}_\alpha\boldsymbol{F}_\alpha+2\frac{\partial^2 f_\alpha}{\partial \boldsymbol{v}_\alpha\,\partial \boldsymbol{r}}:\boldsymbol{v}_\alpha\boldsymbol{F}_\alpha+2\frac{\partial^2 f_\alpha}{\partial \boldsymbol{v}_\alpha\,\partial t}\cdot\boldsymbol{F}_\alpha+2\frac{\partial^2 f_\alpha}{\partial \boldsymbol{r}\,\partial t}\cdot\boldsymbol{v}_\alpha\right\}$$
$$= J_{\alpha,\mathrm{el}}+J_{\alpha,r}, \quad (2.46)$$

where $$J_{\alpha,\mathrm{el}}=\sum_{j=1}^{\mu}\int(f'_\alpha f'_j-f_\alpha f_j)\,P^{\alpha_j}_{\alpha_j}g_{\alpha_j}b\,\mathrm{d}b\,\mathrm{d}\varphi\,\mathrm{d}\boldsymbol{v}_j \quad (2.47)$$

is the usual integral form for elastic collisions of molecules of α-component mixtures and, for example, for bimolecular reactions (Alexeev 1982)

$$A_\alpha+A_\beta\rightleftharpoons A_\gamma+A_\delta,$$

$$J_{\alpha,r}=\tfrac{1}{2}\sum_r\sum_{\beta\gamma\delta}\int(\xi f'_\gamma f'_\delta-f_\alpha f_\beta)\,{}^rP^{\gamma\delta}_{\alpha\beta}g_{\alpha\beta}b\sin\theta\,\mathrm{d}b\,\mathrm{d}\theta\,\mathrm{d}\varphi\,\mathrm{d}\vartheta\,\mathrm{d}\boldsymbol{v}_\beta, \quad (2.48)$$

in usual nominations. We must only notice that

$$\xi=\left(\frac{m_\alpha m_\beta}{m_\gamma m_\delta}\right)^3\frac{s_\alpha s_\beta}{s_\gamma s_\delta}$$

is the factor connected with statistical weight of components.

3. New hydrodynamical equations

(a) *Generalized hydrodynamical equations*

The momentum method must be applied for receiving hydrodynamical equations. As usual in the Enskog theory both sides of the GBE must be multiplied by collision invariants ψ^i_α, connected with conservation laws; after following integration of both sides of this equation over velocity $\boldsymbol{v}_\alpha$, we get the generalized hydrodynamical Enskog equation (GHEE) in the form:

$$\sum_\alpha\int\psi^i_\alpha\left\{\frac{\partial f_\alpha}{\partial t}+\boldsymbol{v}_\alpha\cdot\frac{\partial f_\alpha}{\partial \boldsymbol{r}}+\boldsymbol{F}_\alpha\cdot\frac{\partial f_\alpha}{\partial \boldsymbol{v}_\alpha}\right\}\mathrm{d}\boldsymbol{v}_\alpha-\tau\sum_\alpha\int\psi^i_\alpha\left\{\boldsymbol{F}_\alpha\cdot\frac{\partial f_\alpha}{\partial \boldsymbol{r}}+\frac{\partial \boldsymbol{F}_\alpha}{\partial t}\cdot\frac{\partial f_\alpha}{\partial \boldsymbol{v}_\alpha}+\frac{\partial f_\alpha}{\partial \boldsymbol{v}_\alpha}\boldsymbol{F}_\alpha:\frac{\partial}{\partial \boldsymbol{v}_\alpha}\boldsymbol{F}_\alpha\right.$$
$$+\frac{\partial f_\alpha}{\partial \boldsymbol{v}_\alpha}\boldsymbol{v}_\alpha:\frac{\partial}{\partial \boldsymbol{r}}\boldsymbol{F}_\alpha+\frac{\partial^2 f_\alpha}{\partial t^2}+\frac{\partial^2 f_\alpha}{\partial \boldsymbol{r}\,\partial \boldsymbol{r}}:\boldsymbol{v}_\alpha\boldsymbol{v}_\alpha+\frac{\partial^2 f_\alpha}{\partial \boldsymbol{v}_\alpha\,\partial \boldsymbol{v}_\alpha}:\boldsymbol{F}_\alpha\boldsymbol{F}_\alpha$$
$$\left.+2\frac{\partial^2 f_\alpha}{\partial \boldsymbol{v}_\alpha\,\partial \boldsymbol{r}}:\boldsymbol{v}_\alpha\boldsymbol{F}_\alpha+2\frac{\partial^2 f_\alpha}{\partial \boldsymbol{v}_\alpha\,\partial t}\cdot\boldsymbol{F}_\alpha+2\frac{\partial^2 f_\alpha}{\partial \boldsymbol{r}\,\partial t}\cdot\boldsymbol{v}_\alpha\right\}\mathrm{d}\boldsymbol{v}_\alpha=0. \quad (3.1)$$

The conservation laws of mass, momentum and energy are of course valid in the chemical reactions and therefore for

$$\psi^1_\alpha=m_\alpha,\quad \psi^2_\alpha=m_\alpha\boldsymbol{v}_\alpha,\quad \psi^3_\alpha=\tfrac{1}{2}m_\alpha v^2_\alpha+\epsilon_\alpha \quad (3.2)$$

(ϵ_α is the internal energy of molecule of species α), we can write

$$\sum_\alpha\int\psi^i_\alpha J_{\alpha,\mathrm{el}}\,\mathrm{d}\boldsymbol{v}_\alpha=0,\quad \sum_\alpha\int\psi^i_\alpha J_{\alpha,r}\,\mathrm{d}\boldsymbol{v}_\alpha=0,\quad i=1,2,3. \quad (3.3)$$

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The right-hand side of GHEE (3.1) is zero because of (3.2) and (3.3). We need only to add that the mass conservation law for species α must be discussed separately with the help of relations

$$\int \psi^1_\alpha J_{\alpha,r}\,\mathrm{d}\boldsymbol{v}_\alpha = R_\alpha, \tag{3.4}$$

where R_α is the mass rate of formation of α-species in chemical reactions ($\alpha = 1, \ldots, \mu$).

Now we derive GHEE in the differential form. The derivation mentioned contains very complicated mathematical transformations. Therefore I present here only the principal features of transformations.

(i) *Continuity equation for species α and for mixture*

First of all we introduce the force $\boldsymbol{F}_\alpha$ acting on unit mass as follows

$$\boldsymbol{F}_\alpha = \boldsymbol{F}^{(1)}_\alpha + \boldsymbol{F}^B_\alpha, \tag{3.5}$$

where $\boldsymbol{F}^B_\alpha$ is the Lorentz force, connected with external magnetic field, and $\boldsymbol{F}^{(1)}_\alpha$ are forces of another nature. The first part of the mathematical transformations (the first sum in the left-hand side of (3.1)) is well known (see, for example, Hirschfelder *et al.* 1969) and leads to the usual form of the Enskog continuity equation.

Let us consider for example one of the integral non-classic terms in the second sum of the left-hand side of (3.1):

$$\begin{aligned}\int m_\alpha \frac{\partial^2 f_\alpha}{\mathrm{d}\boldsymbol{v}_\alpha\,\mathrm{d}\boldsymbol{r}} : \boldsymbol{v}_\alpha \boldsymbol{F}_\alpha &= \int m_\alpha \frac{\partial^2 f_\alpha}{\partial \boldsymbol{v}_\alpha\,\partial \boldsymbol{r}} : \boldsymbol{v}_\alpha \boldsymbol{F}^B_\alpha\,\mathrm{d}\boldsymbol{v}_\alpha - \boldsymbol{F}^{(1)}_\alpha \cdot \frac{\partial \rho_\alpha}{\partial \boldsymbol{r}} \\ &= -\frac{q_\alpha}{m_\alpha}\rho_\alpha \bar{\boldsymbol{v}}_\alpha \cdot \operatorname{rot} \boldsymbol{B} - \frac{q_\alpha}{m_\alpha}\frac{\partial}{\partial \boldsymbol{r}} \cdot [\rho_\alpha \bar{\boldsymbol{v}}_\alpha \times \boldsymbol{B}] - \boldsymbol{F}^{(1)}_\alpha \cdot \frac{\partial \rho_\alpha}{\partial \boldsymbol{r}}.\end{aligned} \tag{3.6}$$

In (3.6) we use the usual notations; ρ_α is the density of species α, $\bar{\boldsymbol{v}}_\alpha$ is the mean velocity of α-species molecules,

$$\bar{\boldsymbol{v}}_\alpha = \frac{1}{n_\alpha}\int f_\alpha \boldsymbol{v}_\alpha\,\mathrm{d}\boldsymbol{v}_\alpha, \quad \boldsymbol{v}_0 = \frac{1}{\rho}\sum_\alpha \rho_\alpha \boldsymbol{v}_\alpha. \tag{3.7}$$

In a similar manner we find

$$\begin{aligned}\int m_\alpha \frac{\partial^2 f_\alpha}{\partial \boldsymbol{v}_\alpha\,\partial \boldsymbol{v}_\alpha} : \boldsymbol{F}_\alpha \boldsymbol{F}_\alpha\,\mathrm{d}\boldsymbol{v}_\alpha &= m_\alpha \int \frac{\partial^2 f_\alpha}{\partial \boldsymbol{v}_\alpha\,\partial \boldsymbol{v}_\alpha} : \boldsymbol{F}^B_\alpha \boldsymbol{F}^B_\alpha\,\mathrm{d}\boldsymbol{v}_\alpha \\ &= -2\rho_\alpha \left(\frac{q_\alpha}{m_\alpha}\right)^2 B^2.\end{aligned} \tag{3.8}$$

As a result we obtain the next equation for a mixture:

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \boldsymbol{r}} \cdot (\rho \boldsymbol{v}_0) = \tau \Big\{ &-\sum_\alpha \boldsymbol{F}^{(1)}_\alpha \cdot \frac{\partial \rho_\alpha}{\partial \boldsymbol{r}} - \sum_\alpha \rho_\alpha \frac{\partial}{\partial \boldsymbol{r}} \cdot \boldsymbol{F}^{(1)}_\alpha - \sum_\alpha \frac{q_\alpha}{m_\alpha}\frac{\partial}{\partial \boldsymbol{r}} \cdot (\rho_\alpha \bar{\boldsymbol{v}}_\alpha \times \boldsymbol{B}) \\ &+ \frac{\partial^2 \rho}{\partial t^2} + \frac{\partial^2}{\partial \boldsymbol{r}\,\partial \boldsymbol{r}} : \boldsymbol{P} + \frac{\partial^2}{\partial \boldsymbol{r}\,\partial \boldsymbol{r}} : \rho \boldsymbol{v}_0 \boldsymbol{v}_0 + 2\frac{\partial}{\partial t}\left(\frac{\partial}{\partial \boldsymbol{r}} \cdot \rho \boldsymbol{v}_0\right)\Big\},\end{aligned} \tag{3.9}$$

where tensor $\boldsymbol{P} = \Sigma_\alpha \rho_\alpha \overline{\boldsymbol{V}_\alpha \boldsymbol{V}_\alpha}$; $\boldsymbol{V}_\alpha$ is a peculiar velocity of α-species and the average value $\bar{\varphi}_\alpha$ of any molecular characteristic can be found by

$$\bar{\varphi}_\alpha = \frac{1}{n_\alpha}\int \varphi_\alpha f_\alpha\,\mathrm{d}\boldsymbol{v}_\alpha.$$

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The continuity equation (3.9) has the very interesting form with the additional fluctuation term (proportional to τ and therefore to the Knudsen number) on the right-hand side of this equation. The term on the right-hand side of (3.9) can be considered as the fluctuation term connected with subgrid or microturbulence. The hydrodynamical equations in their usual form are valid for character lengths much more than the mean free path between collisions, and the appearance in hydrodynamical equations of the term proportional to the Knudsen number cannot be considered from this point of view as a breach of the conservation laws. On the other hand the usual form of the Boltzmann equation can be obtained from the BBGKI hierarchy only for the particles considered as material points. For a control volume this kind of particle can be found only inside or outside the volume. This fact leads to the classical form of the continuity equation. For the particles of finite dimensions the third possibility exists when for a time moment the particles are placed on the control surface partly inside and partly outside the surface. As a result we get the fluctuation term in the continuity equation. Introducing now the 'average' values (indicated by $^{\mathrm{a}}$) in the sense of the turbulence theory and neglecting, in the continual limit, the small terms proportional to the product of τ-order terms multiplied by the time and space derivatives of the logarithm of hydrodynamic values, we obtain the divergent form of the continuity equation. Really transformations (3.9) can be written in the divergent form

$$\frac{\partial}{\partial t}\left\{\rho-\tau\left[\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial \boldsymbol{r}}\cdot(\rho \boldsymbol{v}_0)\right]\right\}+\frac{\partial}{\partial \boldsymbol{r}}\cdot\left\{\rho \boldsymbol{v}_0-\tau\left[\frac{\partial}{\partial t}(\rho \boldsymbol{v}_0)+\frac{\partial}{\partial \boldsymbol{r}}\cdot \rho \boldsymbol{v}_0 \boldsymbol{v}_0+\frac{\partial}{\partial \boldsymbol{r}}\cdot \boldsymbol{P}-\sum_\alpha \rho_\alpha \boldsymbol{F}_\alpha^{(1)} -\sum_\alpha \frac{q_\alpha}{m_\alpha}(\rho_\alpha \boldsymbol{v}_0\times \boldsymbol{B}+\boldsymbol{j}_\alpha\times\boldsymbol{B})\right]\right\}=0,$$

where $\boldsymbol{j}_\alpha$ is the diffusional flux of α-species. Therefore

$$\frac{\partial \rho^{\mathrm{a}}}{\partial t}+\frac{\partial}{\partial \boldsymbol{r}}\cdot(\rho \boldsymbol{v}_0)^{\mathrm{a}}=0, \tag{3.10}$$

$$\rho^{\mathrm{a}}=\rho-\tau\left[\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial \boldsymbol{r}}\cdot(\rho \boldsymbol{v}_0)\right], \tag{3.11}$$

$$(\rho \boldsymbol{v}_0)^{\mathrm{a}}=\rho \boldsymbol{v}_0-\tau\left[\frac{\partial}{\partial t}(\rho \boldsymbol{v}_0)+\frac{\partial}{\partial \boldsymbol{r}}\cdot \rho \boldsymbol{v}_0 \boldsymbol{v}_0+\frac{\partial}{\partial \boldsymbol{r}}\cdot \boldsymbol{P}-\sum_\alpha \frac{q_\alpha}{m_\alpha}(\rho_\alpha \boldsymbol{v}_0\times \boldsymbol{B}+\boldsymbol{j}_\alpha\times\boldsymbol{B})-\sum_\alpha \rho_\alpha \boldsymbol{F}_\alpha^{(1)}\right]. \tag{3.12}$$

Relations (3.11) and (3.12) have very interesting structure. If we can state that for any flow the fluctuations of hydrodynamic values disappear, then the values in square brackets in relations (3.11) and (3.12) must be equal to zero. But the square brackets in (3.11) and (3.12) contain the left-hand side of the continuity and momentum equations and hence these values become equal to zero. In conclusion we cite the continuity equation for α-species

$$\frac{\partial \rho_\alpha}{\partial t}+\frac{\partial}{\partial \boldsymbol{r}}\cdot(\rho_\alpha \boldsymbol{v}_0+\boldsymbol{j}_\alpha)=\tau\left\{-\boldsymbol{F}_\alpha^{(1)}\cdot\frac{\partial \rho_\alpha}{\partial \boldsymbol{r}}-\rho_\alpha\frac{\partial}{\partial \boldsymbol{r}}\cdot \boldsymbol{F}_\alpha^{(1)}-\frac{q_\alpha}{m_\alpha}\frac{\partial}{\partial \boldsymbol{r}}\cdot(\rho_\alpha \overline{\boldsymbol{v}}_\alpha\times\boldsymbol{B})+\frac{\partial^2 \rho_\alpha}{\partial t^2} +\frac{\partial^2}{\partial \boldsymbol{r}\,\partial \boldsymbol{r}}:\boldsymbol{P}_\alpha+2\frac{\partial^2}{\partial \boldsymbol{r}\,\partial \boldsymbol{r}}:\boldsymbol{v}_0\boldsymbol{j}_\alpha+\frac{\partial^2}{\partial \boldsymbol{r}\,\partial \boldsymbol{r}}:\rho_\alpha \boldsymbol{v}_0\boldsymbol{v}_0+2\frac{\partial}{\partial t}\left(\frac{\partial}{\partial \boldsymbol{r}}\cdot \rho_\alpha \boldsymbol{v}_0\right)+2\frac{\partial}{\partial t}\left(\frac{\partial}{\partial \boldsymbol{r}}\cdot \boldsymbol{j}_\alpha\right)\right\}+R_\alpha. \tag{3.13}$$

(ii) *Momentum equation for multicomponent mixture*

The momentum equation for the multicomponent mixture can be received from GHEE (3.1) using as the collision invariant $\psi_\alpha^2 = m_\alpha \boldsymbol{V}_\alpha$ or $\psi_\alpha^2 = m_\alpha \boldsymbol{v}_\alpha$. Both forms of the equation are equivalent and can be transformed from one to another with the help of linear combination of hydrodynamical equations for mass, momentum and energy.

As earlier, we point out only transformations of terms we need; for example

$$\sum_\alpha \int m_\alpha \boldsymbol{v}_\alpha \frac{\partial^2 f_\alpha}{\partial \boldsymbol{v}_\alpha \partial \boldsymbol{r}} : \boldsymbol{v}_\alpha \boldsymbol{F}_\alpha \, \mathrm{d}\boldsymbol{v}_\alpha = -\sum_\alpha \boldsymbol{F}_\alpha^{(1)} \frac{\partial}{\partial \boldsymbol{r}} \cdot (\rho_\alpha \overline{\boldsymbol{v}}_\alpha) - \sum_\alpha \left(\boldsymbol{F}_\alpha^{(1)} \cdot \frac{\partial}{\partial \boldsymbol{r}} \right) (\rho_\alpha \boldsymbol{v}_\alpha)$$

$$- \left[\frac{\partial}{\partial \boldsymbol{r}} \cdot \sum_\alpha \frac{q_\alpha}{m_\alpha} \rho_\alpha \overline{\boldsymbol{v}_\alpha \boldsymbol{v}_\alpha} \right] \times \boldsymbol{B} - \left(\boldsymbol{B} \times \frac{\partial}{\partial \boldsymbol{r}} \right) \cdot \sum_\alpha \left(\frac{q_\alpha}{m_\alpha} \right) \rho_\alpha \overline{\boldsymbol{v}_\alpha \boldsymbol{v}_\alpha}, \tag{3.14}$$

$$\sum_\alpha \int m_\alpha \boldsymbol{v}_\alpha \left(\boldsymbol{F}_\alpha \cdot \frac{\partial f_\alpha}{\partial \boldsymbol{r}} \right) \mathrm{d}\boldsymbol{v}_\alpha = \sum_\alpha \left(\boldsymbol{F}_\alpha^{(1)} \cdot \frac{\partial}{\partial \boldsymbol{r}} \right) (\rho_\alpha \overline{\boldsymbol{v}}_\alpha) + \sum_\alpha \left(\frac{q_\alpha}{m_\alpha} \right) \boldsymbol{B} \cdot \frac{\partial}{\partial \boldsymbol{r}} \times (\boldsymbol{v}_\alpha \boldsymbol{v}_\alpha \rho_\alpha). \tag{3.15}$$

In the last relation we introduce the tensor of the type $(\partial/\partial \boldsymbol{r}) \times \boldsymbol{AA}$ (vector product of $\partial/\partial \boldsymbol{r}$ and dyad $\boldsymbol{AA}$). Therefore the scalar product of this tensor and vector leads to the vector whose components can be written in the form

$$\left[\boldsymbol{B} \cdot \left(\frac{\partial}{\partial \boldsymbol{r}} \times (\overline{\boldsymbol{v}_\alpha \boldsymbol{v}_\alpha} \rho_\alpha) \right) \right]_i = \boldsymbol{B} \cdot \left(\frac{\partial}{\partial \boldsymbol{r}} \times \int f_\alpha \boldsymbol{v}_\alpha v_{\alpha i} \, \mathrm{d}\boldsymbol{v}_\alpha \right) \quad (i = 1, 2, 3). \tag{3.16}$$

As a result, we have the following momentum equations:

$$\frac{\partial}{\partial t} (\rho \boldsymbol{v}_0) + \frac{\partial}{\partial \boldsymbol{r}} \cdot (\rho \boldsymbol{v}_0 \boldsymbol{v}_0) + \frac{\partial}{\partial \boldsymbol{r}} \cdot \boldsymbol{P} - \sum_\alpha \rho_\alpha \boldsymbol{F}_\alpha^{(1)} - \sum_\alpha \frac{q_\alpha}{m_\alpha} \rho_\alpha (\overline{\boldsymbol{v}}_\alpha \times \boldsymbol{B})$$

$$= \tau \left\{ \sum_\alpha \frac{q_\alpha}{m_\alpha} \boldsymbol{B} \cdot \frac{\partial}{\partial \boldsymbol{r}} \times (\boldsymbol{v}_\alpha \boldsymbol{v}_\alpha \rho_\alpha) - \sum_\alpha \rho_\alpha \frac{\partial \boldsymbol{F}_\alpha^{(1)}}{\partial t} + \frac{\partial \boldsymbol{B}}{\partial t} \times \sum_\alpha q_\alpha n_\alpha \overline{\boldsymbol{v}}_\alpha - \sum_\alpha \rho_\alpha \overline{\boldsymbol{v}}_\alpha \frac{\partial}{\partial \boldsymbol{r}} \cdot \boldsymbol{F}_\alpha^{(1)} \right.$$

$$+ \sum_\alpha q_\alpha n_\alpha \overline{\left(\boldsymbol{v}_\alpha \cdot \frac{\partial}{\partial \boldsymbol{r}} \right) \boldsymbol{B} \times \boldsymbol{v}_\alpha} + \sum_\alpha \frac{q_\alpha}{m_\alpha} \rho_\alpha \overline{\boldsymbol{v}_\alpha \boldsymbol{v}_\alpha} \cdot \operatorname{rot} \boldsymbol{B} + \frac{\partial^2}{\partial t^2} (\rho \boldsymbol{v}_0) + \frac{\partial^2}{\partial \boldsymbol{r} \, \partial \boldsymbol{r}} : (\boldsymbol{P} + \rho \boldsymbol{v}_0 \boldsymbol{v}_0) \boldsymbol{v}_0$$

$$+ 2 \frac{\partial^2}{\partial \boldsymbol{r} \, \partial \boldsymbol{r}} : \sum_\alpha \rho_\alpha \overline{(\boldsymbol{V}_\alpha \boldsymbol{v}_0) \boldsymbol{V}_\alpha} - \sum_\alpha \rho_\alpha \left(\overline{\boldsymbol{v}}_\alpha \cdot \frac{\partial}{\partial \boldsymbol{r}} \right) \boldsymbol{F}_\alpha^{(1)} + \frac{\partial^2}{\partial \boldsymbol{r} \, \partial \boldsymbol{r}} : \sum_\alpha \rho_\alpha \overline{(\boldsymbol{V}_\alpha \boldsymbol{V}_\alpha) \boldsymbol{V}_\alpha}$$

$$+ \sum_\alpha \left\{ \rho_\alpha \left(\frac{q_\alpha}{m_\alpha} \right)^2 [\boldsymbol{B} (\boldsymbol{B} \cdot \overline{\boldsymbol{v}}_\alpha) - B^2 \overline{\boldsymbol{v}}_\alpha] \right\} - 2 \sum_\alpha \boldsymbol{F}_\alpha^{(1)} \frac{\partial}{\partial \boldsymbol{r}} \cdot (\rho_\alpha \overline{\boldsymbol{v}}_\alpha) - \sum_\alpha \left(\boldsymbol{F}_\alpha^{(1)} \cdot \frac{\partial}{\partial \boldsymbol{r}} \right) (\rho_\alpha \overline{\boldsymbol{v}}_\alpha)$$

$$- 2 \left[\frac{\partial}{\partial \boldsymbol{r}} \cdot \sum_\alpha \frac{q_\alpha}{m_\alpha} \rho_\alpha \overline{\boldsymbol{v}_\alpha \boldsymbol{v}_\alpha} \right] \times \boldsymbol{B} - 2 \left(\boldsymbol{B} \times \frac{\partial}{\partial \boldsymbol{r}} \right) \cdot \sum_\alpha \frac{q_\alpha}{m_\alpha} \rho_\alpha \overline{\boldsymbol{v}_\alpha \boldsymbol{v}_\alpha} - \sum_\alpha q_\alpha n_\alpha (\boldsymbol{F}_\alpha^{(1)} \times \boldsymbol{B})$$

$$\left. - 2 \sum_\alpha \boldsymbol{F}_\alpha^{(1)} \frac{\partial \rho_\alpha}{\partial t} - 2 \left[\frac{\partial}{\partial t} \sum_\alpha \frac{q_\alpha}{m_\alpha} \rho_\alpha \overline{\boldsymbol{v}}_\alpha \right] \times \boldsymbol{B} + 2 \frac{\partial^2}{\partial \boldsymbol{r} \, \partial t} \cdot \rho \boldsymbol{v}_0 \boldsymbol{v}_0 + 2 \frac{\partial^2}{\partial \boldsymbol{r} \, \partial t} \cdot \boldsymbol{P} \right\}. \tag{3.17}$$

(iii) *Energy equation for a multicomponent mixture with chemical reactions*

As in the previous case the energy equation will be written for the collision invariant, which is connected with the particle velocity:

$$\psi_\alpha^3 = \tfrac{1}{2} m_\alpha v_\alpha^2 + \epsilon_\alpha.$$

The use of the collision invariant $\frac{1}{2}m_\alpha V_\alpha^2+\epsilon_\alpha$ leads to the linear dependent equation. In common cases the generalized energy equation has a rather large form. Nevertheless, we write this equation for the case when the external magnetic field is absent ($\mathbf{B}=0$):

$$\begin{aligned}
&\frac{\partial}{\partial t}(\tfrac{1}{2}\rho v_0^2)+\frac{3}{2}\frac{\partial p}{\partial t}+\frac{\partial}{\partial t}\sum_\alpha n_\alpha\epsilon_\alpha+\frac{1}{2}\frac{\partial}{\partial \mathbf{r}}\cdot(\mathbf{v}_0\rho v_0^2)+\frac{\partial}{\partial \mathbf{r}}\cdot(\mathbf{v}_0\cdot\mathbf{P})+\frac{3}{2}\frac{\partial}{\partial \mathbf{r}}\cdot(\mathbf{v}_0 p)\\
&\qquad+\frac{\partial}{\partial \mathbf{r}}\cdot(\mathbf{v}_0\sum_\alpha n_\alpha\epsilon_\alpha)+\frac{\partial}{\partial \mathbf{r}}\cdot\mathbf{q}-\mathbf{v}_0\sum_\alpha\rho_\alpha\mathbf{F}_\alpha^{(1)}-\sum_\alpha\mathbf{F}_\alpha^{(1)}\cdot\mathbf{j}_\alpha\\
&=\tau\Big\{-\sum_\alpha\frac{\partial\mathbf{F}_\alpha^{(1)}}{\partial t}\rho_\alpha\bar{\mathbf{v}}_\alpha-\sum_\alpha[\tfrac{1}{2}\rho_\alpha v_0^2+\mathbf{j}_\alpha\cdot\mathbf{v}_0+\tfrac{3}{2}p_\alpha+\epsilon_\alpha n_\alpha]\frac{\partial}{\partial \mathbf{r}}\cdot\mathbf{F}_\alpha^{(1)}-\sum_\alpha\rho_\alpha\overline{\mathbf{v}_\alpha\mathbf{v}_\alpha}:\frac{\partial}{\partial \mathbf{r}}\mathbf{F}_\alpha^{(1)}\\
&\quad+\frac{\partial^2}{\partial t^2}[\tfrac{1}{2}\rho v_0^2+\tfrac{3}{2}p+\sum_\alpha\epsilon_\alpha n_\alpha]+\frac{\partial^2}{\partial \mathbf{r}\,\partial \mathbf{r}}:(\mathbf{v}_0\mathbf{v}_0\tfrac{1}{2}\rho v_0^2)+\frac{1}{2}\frac{\partial^2}{\partial \mathbf{r}\,\partial \mathbf{r}}:v_0^2\mathbf{P}+2\frac{\partial^2}{\partial \mathbf{r}\,\partial \mathbf{r}}:[(\mathbf{v}_0\cdot\mathbf{P})\mathbf{v}_0]\\
&\quad+\frac{\partial^2}{\partial \mathbf{r}\,\partial \mathbf{r}}:\sum_\alpha(\mathbf{v}_0\cdot\overline{\mathbf{V}_\alpha)\rho_\alpha\mathbf{V}_\alpha\mathbf{V}_\alpha}+\frac{1}{2}\frac{\partial^2}{\partial \mathbf{r}\,\partial \mathbf{r}}:\sum_\alpha\rho_\alpha\overline{\mathbf{V}_\alpha\mathbf{V}_\alpha V_\alpha^2}+\frac{3}{2}\frac{\partial^2}{\partial \mathbf{r}\,\partial \mathbf{r}}:p\mathbf{v}_0\mathbf{v}_0+2\frac{\partial^2}{\partial \mathbf{r}\,\partial \mathbf{r}}:\mathbf{v}_0\mathbf{q}\\
&\quad+\frac{\partial^2}{\partial \mathbf{r}\,\partial \mathbf{r}}:\sum_\alpha\frac{\epsilon_\alpha}{m_\alpha}[\rho_\alpha\mathbf{v}_0\mathbf{v}_0+\mathbf{P}_\alpha]+\sum_\alpha\rho_\alpha\mathbf{F}_\alpha^{(1)2}-2\sum_\alpha\mathbf{F}_\alpha^{(1)}\cdot\left[\frac{\partial}{\partial \mathbf{r}}\cdot(\rho_\alpha\mathbf{v}_0\mathbf{v}_0+\mathbf{v}_0\mathbf{j}_\alpha+\mathbf{j}_\alpha\mathbf{v}_0+\mathbf{P}_\alpha)\right]\\
&\quad-\sum_\alpha\left(\mathbf{F}_\alpha^{(1)}\cdot\frac{\partial}{\partial \mathbf{r}}\right)[\epsilon_\alpha n_\alpha+\tfrac{3}{2}p_\alpha+\mathbf{v}_0\cdot\mathbf{j}_\alpha+\tfrac{1}{2}\rho_\alpha v_0^2]-2\sum_\alpha\mathbf{F}_\alpha^{(1)}\cdot\frac{\partial}{\partial t}(\rho_\alpha\mathbf{v}_0)-2\sum_\alpha\mathbf{F}_\alpha^{(1)}\cdot\frac{\partial\mathbf{j}_\alpha}{\partial t}\\
&\quad+2\frac{\partial^2}{\partial \mathbf{r}\,\partial t}\cdot[\mathbf{v}_0(\tfrac{1}{2}\rho v_0^2+\tfrac{3}{2}p+\sum_\alpha\epsilon_\alpha n_\alpha)]+2\frac{\partial^2}{\partial \mathbf{r}\,\partial t}\cdot(\mathbf{v}_0\cdot\mathbf{P})+2\frac{\partial^2}{\partial \mathbf{r}\,\partial t}\cdot\mathbf{q}\Big\},
\end{aligned}\tag{3.18}$$

where $\mathbf{q}$ is the energy-flux vector and p_α, p are the partial and total pressure respectively. The partial forms and particular cases of these generalized hydrodynamical equations will be discussed in the following sections.

(*b*) *Method of the* GBE *solution*

Let us consider the generalized Boltzmann equation

$$\frac{\mathscr{D}f_\alpha}{\mathscr{D}t}-\tau\frac{\mathscr{D}}{\mathscr{D}t}\frac{\mathscr{D}f_\alpha}{\mathscr{D}t}=\sum_{j=1}^{\mu}\int[f'_\alpha f'_j-f_\alpha f_j]\,g_{\alpha j}\,b\,\mathrm{d}b\,\mathrm{d}\varphi\,\mathrm{d}\mathbf{v}_j,\tag{3.19}$$

written for the multicomponent mixture of non-reacting gases. For simplicity we introduce the condition that all external forces are absent ($\mathbf{F}_\alpha=0$). Using the main ideas of the classical Chapman–Enskog method we construct the solution of (3.19) expanding the dimensionless distribution function in power series in the Knudsen number. In dimensional form we have

$$f_\alpha=f_\alpha^{(0)}+f_\alpha^{(1)},\tag{3.20}$$

where $f_\alpha^{(0)}$ as usual is the maxwellian function. For the first approximation we have

$$\begin{aligned}
&\frac{\mathscr{D}f_\alpha^{(0)}}{\mathscr{D}t}-\tau\frac{\partial}{\partial t}\frac{\mathscr{D}f_\alpha^{(0)}}{\mathscr{D}t}-\tau\mathbf{v}_\alpha\cdot\frac{\partial}{\partial \mathbf{r}}\frac{\mathscr{D}f_\alpha^{(0)}}{\mathscr{D}t}\\
&\qquad=\sum_{j=1}^{\mu}\int[f_\alpha^{(0)\prime}f_j^{(1)\prime}+f_\alpha^{(1)\prime}f_j^{(0)\prime}-f_\alpha^{(0)}f_j^{(1)}-f_\alpha^{(1)}f_j^{(0)}]\,g_{\alpha j}\,b\,\mathrm{d}b\,\mathrm{d}\varphi\,\mathrm{d}\mathbf{v}_j,
\end{aligned}\tag{3.21}$$

where
$$\frac{\mathscr{D}f_\alpha^{(0)}}{\mathscr{D}t}=\frac{\partial f_\alpha^{(0)}}{\partial t}+\boldsymbol{v}_\alpha\cdot\frac{\partial f_\alpha^{(0)}}{\partial \boldsymbol{r}}. \tag{3.22}$$

We search for the solution of (3.21) in the form

$$f_\alpha^{(1)}=f_\alpha^{(0)}\phi_\alpha-\tau\frac{\partial}{\partial t}(f_\alpha^{(0)}\phi_\alpha)-\tau\boldsymbol{v}_\alpha\cdot\frac{\partial}{\partial \boldsymbol{r}}(f_\alpha^{(0)}\phi_\alpha). \tag{3.23}$$

In (3.23), ϕ_α is the solution, received by Chapman and Enskog for the equation

$$\frac{\mathscr{D}f_\alpha^{(0)}}{\mathscr{D}t}=\sum_j\int f_\alpha^{(0)}f_j^{(0)}(\phi_\alpha+\phi_j-\phi_\alpha-\phi_j)\,g_{\alpha j}\,b\,\mathrm{d}b\,\mathrm{d}\varphi\,\mathrm{d}\boldsymbol{v}_j. \tag{3.24}$$

Using (3.23) and the linearity of (3.21) we reduce this equation to the system of equations:

$$\frac{\mathscr{D}f_\alpha^{(0)}}{\mathscr{D}t}=\sum_j\int f_\alpha^{(0)}f_j^{(0)}(\phi'_\alpha+\phi'_j-\phi_\alpha-\phi_j)\,g_{\alpha j}\,b\,\mathrm{d}b\,\mathrm{d}\varphi\,\mathrm{d}\boldsymbol{v}_j, \tag{3.25}$$

$$\frac{\partial}{\partial t}\frac{\mathscr{D}f_\alpha^{(0)}}{\mathscr{D}t}=\frac{\partial}{\partial t}\sum_j\int f_\alpha^{(0)}f_j^{(0)}(\phi'_\alpha+\phi'_j-\phi_\alpha-\phi_j)\,g_{\alpha j}\,b\,\mathrm{d}b\,\mathrm{d}\varphi\,\mathrm{d}\boldsymbol{v}_j, \tag{3.26}$$

$$\frac{\partial}{\partial \boldsymbol{r}}\cdot\left(\boldsymbol{v}_0\frac{\mathscr{D}f_\alpha^{(0)}}{\mathscr{D}t}\right)=\frac{\partial}{\partial \boldsymbol{r}}\cdot\left[\boldsymbol{v}_0\sum_j\int f_\alpha^{(0)}f_j^{(0)}(\phi'_\alpha+\phi'_j-\phi_\alpha-\phi_j)\,g_{\alpha j}\,b\,\mathrm{d}b\,\mathrm{d}\varphi\,\mathrm{d}\boldsymbol{v}_j\right], \tag{3.27}$$

$$\frac{\partial}{\partial \boldsymbol{r}}\cdot\left(\boldsymbol{V}_\alpha\frac{\mathscr{D}f_\alpha^{(0)}}{\mathscr{D}t}\right)=\frac{\partial}{\partial \boldsymbol{r}}\cdot\sum_j\int f_\alpha^{(0)}f_j^{(0)}(\boldsymbol{V}'_\alpha\phi'_\alpha+\boldsymbol{V}'_j\phi'_j-\boldsymbol{V}_\alpha\phi_\alpha-\boldsymbol{V}_j\phi_j)\,g_{\alpha j}\,b\,\mathrm{d}b\,\mathrm{d}\varphi\,\mathrm{d}\boldsymbol{v}_j. \tag{3.28}$$

Equation (3.25) coincides with (3.24); the solution of this equation is well known (Chapman & Cowling 1939). I should emphasize that the substantial derivative $\mathscr{D}f_\alpha^{(0)}/\mathscr{D}t$ must be calculated with $\tau=0$. Obviously there are no contradictory equations among (3.25)–(3.27). As to (3.28), note that the use of the momentum method of the (3.28) solution leads to the identical zero in both sides of the equations. Therefore the solution of (3.21) can be realized with the help of the traditional Chapman–Enskog method. I intend to keep in the right-hand sides of generalized hydrodynamical equations only the terms which are proportional to τ. The pressure tensor, $\boldsymbol{P}$, the diffusional flux, $\boldsymbol{j}_\alpha$ of species, α, and energy flux, $\boldsymbol{q}_\alpha$ in the right-hand sides of the GHE must be calculated in the zero τ-approximation, using as correction for $f_\alpha^{(0)}$ only the function $f_\alpha^{(0)}\phi_\alpha$. Any values obtained this way will be marked as $\boldsymbol{P}\hat{}$.

I begin with the continuity equation

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial \boldsymbol{r}}\cdot(\rho\boldsymbol{v}_0)=\tau\left[\frac{\partial^2\rho}{\partial t^2}+\frac{\partial^2}{\partial\boldsymbol{r}\,\partial\boldsymbol{r}}:\boldsymbol{P}\hat{}+\frac{\partial^2}{\partial\boldsymbol{r}\,\partial\boldsymbol{r}}:\rho\boldsymbol{v}_0\,\boldsymbol{v}_0+2\frac{\partial^2}{\partial t\,\partial\boldsymbol{r}}\cdot(\rho\boldsymbol{v}_0)\right]. \tag{3.29}$$

The momentum equation can be written as

$$\frac{\partial}{\partial t}(\rho\boldsymbol{v}_0)+\boldsymbol{v}_0\left(\frac{\partial}{\partial\boldsymbol{r}}\cdot\rho\boldsymbol{v}_0\right)+\left(\rho\boldsymbol{v}_0\cdot\frac{\partial}{\partial\boldsymbol{r}}\right)\boldsymbol{v}_0+\frac{\partial}{\partial\boldsymbol{r}}\cdot\boldsymbol{P}=\tau\left[\frac{\partial^2}{\partial t^2}(\rho\boldsymbol{v}_0)+\frac{\partial^2}{\partial\boldsymbol{r}\,\partial\boldsymbol{r}}:(\boldsymbol{P}\hat{}+\rho\boldsymbol{v}_0\,\boldsymbol{v}_0)\,\boldsymbol{v}_0\right.$$
$$\left.+2\frac{\partial^2}{\partial\boldsymbol{r}\,\partial\boldsymbol{r}}:\sum_\alpha\rho_\alpha\overline{(\boldsymbol{V}_\alpha\boldsymbol{v}_0)\,\boldsymbol{V}_\alpha}+\frac{\partial^2}{\partial\boldsymbol{r}\,\partial\boldsymbol{r}}:\sum_\alpha\rho_\alpha\overline{(\boldsymbol{V}_\alpha\boldsymbol{V}_\alpha)\,\boldsymbol{V}_\alpha}+2\frac{\partial^2}{\partial t\,\partial\boldsymbol{r}}\cdot\rho\boldsymbol{v}_0\,\boldsymbol{v}_0+2\frac{\partial^2}{\partial t\,\partial\boldsymbol{r}}\cdot\boldsymbol{P}\hat{}\right]. \tag{3.30}$$

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For the pressure tensor $\boldsymbol{P}$ one obtains

$$\boldsymbol{P} = \sum_{\alpha}\Big\{m_\alpha \int \boldsymbol{V}_\alpha \boldsymbol{V}_\alpha f_\alpha^{(0)}\,\mathrm{d}\boldsymbol{v}_\alpha + m_\alpha \int \boldsymbol{V}_\alpha \boldsymbol{V}_\alpha f_\alpha^{(0)}\phi_\alpha\,\mathrm{d}\boldsymbol{v}_\alpha$$

$$-\tau m_\alpha \int \boldsymbol{V}_\alpha \boldsymbol{V}_\alpha \frac{\partial}{\partial t}(f_\alpha^{(0)}\phi_\alpha)\,\mathrm{d}\boldsymbol{v}_\alpha + m_\alpha \int \boldsymbol{V}_\alpha \boldsymbol{V}_\alpha \frac{\partial}{\partial \boldsymbol{r}}\cdot(\boldsymbol{v}_\alpha f_\alpha^{(0)}\phi_\alpha)\,\mathrm{d}\boldsymbol{v}_\alpha\Big\} = \sum_\alpha \boldsymbol{P}_\alpha, \quad (3.31)$$

$$\boldsymbol{P} = p\boldsymbol{I} - 2\eta\boldsymbol{S} - \tau\Big\{-2\frac{\partial}{\partial t}(\boldsymbol{S}\eta) - 2\Big(\frac{\partial}{\partial \boldsymbol{r}}\cdot\Big[\boldsymbol{v}_0\Big)\eta\boldsymbol{S}\Big] + 2\boldsymbol{v}_0\frac{\partial}{\partial \boldsymbol{r}}\cdot(\eta\boldsymbol{S})$$

$$-2\eta\Big(\boldsymbol{S}\cdot\frac{\partial}{\partial \boldsymbol{r}}\Big)\boldsymbol{v}_0 - \boldsymbol{T} + \boldsymbol{D} + {}^{v_0}\boldsymbol{H}\Big\}, \quad (3.32)$$

where we introduce the tensors: $\boldsymbol{I}$, the unit tensor; $\boldsymbol{T}$, which has the components $(T = \Sigma_\alpha \boldsymbol{T}_\alpha)$

$$T_{kl,\alpha} = \frac{\partial}{\partial r_k}\Big(k_\alpha \frac{\partial T}{\partial r_l}\Big) + \frac{\partial}{\partial r_l}\Big(k_\alpha \frac{\partial T}{\partial r_k}\Big) + \delta_{kl}\sum_{i=1}^{3}\frac{\partial}{\partial r_i}\Big(k_\alpha \frac{\partial T}{\partial r_i}\Big), \quad (3.33)$$

where T is the temperature of mixture and $k_\alpha = (k/m_\alpha)D_\alpha^T + \frac{2}{5}\lambda'_\alpha$. D_α^T is the thermal diffusion coefficient; k is the Boltzmann constant and λ'_α is thermal conductivity;

$$\lambda'_\alpha = -\tfrac{5}{4}kn_\alpha\sqrt{\frac{2kT}{m_\alpha}}\,a_{\alpha 1}.$$

$a_{\alpha 1}$ is the coefficient in the Sonine expansion.

$\boldsymbol{D}$ is the tensor for which $D = \Sigma_\alpha \boldsymbol{D}_\alpha$. The tensor $\boldsymbol{D}$ has components

$$D_{kl,\alpha} = \frac{\partial}{\partial r_k}l_{\alpha l} + \frac{\partial}{\partial r_l}l_{\alpha k} + \delta_{kl}\sum_{i=1}^{3}\frac{\partial}{\partial r_i}l_{\alpha i}, \quad (3.34)$$

where

$$l_{\alpha k} = \frac{p}{\rho}n\sum_\beta m_\beta(D_{\alpha\beta} - D^1_{\alpha\beta}).$$

Relation (3.34) contains usual diffusion coefficient $D_{\alpha\beta}$:

$$D_{\alpha\beta} = \frac{\rho n_\alpha}{2nm_\beta}\sqrt{\frac{2kT}{m_\alpha}}\,c_{\alpha 0}^{(\beta,\alpha)}, \quad (3.35)$$

and the 'first' diffusional coefficient

$$D^1_{\alpha\beta} = \frac{\rho n_\alpha}{2nm_\beta}\sqrt{\frac{2kT}{m_\alpha}}\,c_{\alpha 1}^{(\beta,\alpha)}. \quad (3.36)$$

$c_{\alpha 0}^{(\beta,\alpha)}, c_{\alpha 1}^{(\beta,\alpha)}$ are coefficients in the Sonine expansion.

Tensor ${}^{v_0}\boldsymbol{H} = \Sigma_\alpha {}^{v_0}\boldsymbol{H}_\alpha$ has components.

$${}^{v_0}H_{kl,\alpha} = -2\sum_{i=1}^{3}\frac{\partial}{\partial r_i}\Big\{v_{0k}\,\eta_\alpha\Big[\tfrac{1}{2}\Big(\frac{\partial v_{0l}}{\partial r_i} + \frac{\partial v_{0i}}{\partial r_l}\Big) - \tfrac{1}{3}\delta_{il}\frac{\partial}{\partial \boldsymbol{r}}\cdot\boldsymbol{v}_0\Big]\Big\}. \quad (3.37)$$

$\boldsymbol{S}$ is the usual stress tensor. The cross-disposition of brackets in (3.32) underlines the order of calculations.

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The right-hand side of (3.30) contains two averaged terms, which can be calculated owing to $f_\alpha = f_\alpha^{(0)}(1+\phi_\alpha)$:

$$\frac{\partial^2}{\partial \boldsymbol{r}\,\partial \boldsymbol{r}}:\sum_\alpha \rho_\alpha \overline{(\boldsymbol{V}_\alpha \boldsymbol{v}_0)\,\boldsymbol{V}_\alpha} = -2\frac{\partial^2}{\partial \boldsymbol{r}\,\partial \boldsymbol{r}}:\boldsymbol{v}_0\,\eta\boldsymbol{S} + \frac{\partial}{\partial \boldsymbol{r}}\left[\frac{\partial}{\partial \boldsymbol{r}}\cdot(p\boldsymbol{v}_0)\right], \tag{3.38}$$

$$\frac{\partial^2}{\partial \boldsymbol{r}\,\partial \boldsymbol{r}}:\sum_\alpha \rho_\alpha \overline{(\boldsymbol{V}_\alpha \boldsymbol{V}_\alpha)\,\boldsymbol{V}_\alpha} = -2\frac{\partial^2}{\partial \boldsymbol{r}\,\partial \boldsymbol{r}}\cdot\left(\check{k}\frac{\partial T}{\partial \boldsymbol{r}}\right) - \Delta\left(\check{k}\frac{\partial T}{\partial \boldsymbol{r}}\right)$$

$$+2\frac{\partial^2}{\partial \boldsymbol{r}\,\partial \boldsymbol{r}}\cdot\left[\frac{pn}{\rho}\sum_{\alpha,\beta} m_\beta(D_{\alpha\beta}-D^1_{\alpha\beta})\,\boldsymbol{d}_\beta\right] + \Delta\left[\frac{pn}{\rho}\sum_{\alpha\beta} m_\beta(D_{\alpha\beta}-D^1_{\alpha\beta})\,\boldsymbol{d}_\beta\right], \tag{3.39}$$

where
$$\boldsymbol{d}_\beta = \frac{\partial}{\partial \boldsymbol{r}}\left(\frac{n_\beta}{n}\right) + \left(\frac{n_\beta}{n} - \frac{\rho_\beta}{\rho}\right)\frac{\partial \ln p}{\partial \boldsymbol{r}} \quad \text{and} \quad \check{k} = \sum_{\alpha=1}^{\mu} k_\alpha.$$

Note (see (3.31)–(3.37)) that in the common case the tensor $\boldsymbol{P}$ contains the terms connected with processes of diffusion, thermodiffusion and thermal conductivity.

We can now proceed to derive the conservation energy in the generalized Navier–Stokes approximation. Using the collision invariant $\psi_\alpha^3 = \tfrac{1}{2}m_\alpha V_\alpha^2$ it is possible to write the energy equation in the form:

$$\frac{\partial}{\partial t}(\rho\tilde{U}) + \frac{\partial}{\partial \boldsymbol{r}}\cdot(\rho\boldsymbol{v}_0\tilde{U}) + \frac{\partial}{\partial \boldsymbol{r}}\cdot\boldsymbol{q} + \boldsymbol{P}:\frac{\partial}{\partial \boldsymbol{r}}\boldsymbol{v}_0$$

$$= \tau\left\{\frac{\partial^2}{\partial t^2}(\rho\tilde{U}) + \rho\left(\frac{\partial \boldsymbol{v}_0}{\partial t}\right)^2 + \frac{1}{2}\frac{\partial^2}{\partial \boldsymbol{r}\,\partial \boldsymbol{r}}:v_0^2\boldsymbol{P}\hat{} + \frac{\partial^2}{\partial \boldsymbol{r}\,\partial \boldsymbol{r}}:\sum_\alpha \rho_\alpha\overline{(\boldsymbol{V}_\alpha \boldsymbol{V}_\alpha)\,(\boldsymbol{V}_\alpha\cdot\boldsymbol{v}_0)}\right.$$

$$+\frac{1}{2}\frac{\partial^2}{\partial \boldsymbol{r}\,\partial \boldsymbol{r}}:\sum_\alpha \rho_\alpha\overline{(\boldsymbol{V}_\alpha \boldsymbol{V}_\alpha)\,V_\alpha^2} + 2\frac{\partial^2}{\partial \boldsymbol{r}\,\partial \boldsymbol{r}}:\sum_\alpha \rho_\alpha\overline{(\boldsymbol{V}_\alpha \boldsymbol{v}_0)\,\boldsymbol{V}_\alpha\cdot\boldsymbol{v}_0} + \frac{\partial^2}{\partial \boldsymbol{r}\,\partial \boldsymbol{r}}:\sum_\alpha \rho_\alpha\overline{(\boldsymbol{V}_\alpha \boldsymbol{v}_0)\,V_\alpha^2}$$

$$+\frac{1}{2}\frac{\partial^2}{\partial \boldsymbol{r}\,\partial \boldsymbol{r}}:\rho(\boldsymbol{v}_0\boldsymbol{v}_0)\,v_0^2 + \frac{3}{2}\frac{\partial^2}{\partial \boldsymbol{r}\,\partial \boldsymbol{r}}:p(\boldsymbol{v}_0\boldsymbol{v}_0) - \boldsymbol{v}_0\cdot\frac{\partial^2}{\partial \boldsymbol{r}\,\partial \boldsymbol{r}}:(\boldsymbol{P}\hat{} + \rho\boldsymbol{v}_0\boldsymbol{v}_0)\,\boldsymbol{v}_0$$

$$-2\boldsymbol{v}_0\cdot\frac{\partial^2}{\partial \boldsymbol{r}\,\partial \boldsymbol{r}}:\sum_\alpha \rho_\alpha\overline{(\boldsymbol{V}_\alpha \boldsymbol{v}_0)\,\boldsymbol{V}_\alpha} - \boldsymbol{v}_0\cdot\frac{\partial^2}{\partial \boldsymbol{r}\,\partial \boldsymbol{r}}:\sum_\alpha \rho_\alpha\overline{(\boldsymbol{V}_\alpha \boldsymbol{V}_\alpha)\,\boldsymbol{V}_\alpha} + \tfrac{1}{2}v_0^2\frac{\partial^2}{\partial \boldsymbol{r}\,\partial \boldsymbol{r}}:\boldsymbol{P}\hat{}$$

$$+\tfrac{1}{2}v_0^2\frac{\partial^2}{\partial \boldsymbol{r}\,\partial \boldsymbol{r}}:\rho\boldsymbol{v}_0\boldsymbol{v}_0 + 2\frac{\partial^2}{\partial \boldsymbol{r}\,\partial t}\cdot[\boldsymbol{q}\hat{} + \boldsymbol{v}_0\cdot\boldsymbol{P}\hat{} + \tfrac{3}{2}p\boldsymbol{v}_0 + \tfrac{1}{2}\rho v_0^2\boldsymbol{v}_0]$$

$$\left.-2\boldsymbol{v}_0\cdot\left[\frac{\partial^2}{\partial \boldsymbol{r}\,\partial t}\cdot(\boldsymbol{P}\hat{} + \rho\boldsymbol{v}_0\boldsymbol{v}_0)\right] + v_0^2\frac{\partial^2}{\partial \boldsymbol{r}\,\partial t}\cdot(\rho\boldsymbol{v}_0\right\}, \tag{3.40}$$

with
$$\rho\tilde{V} = \tfrac{3}{2}nkT.$$

To use (3.40) we need calculate the heat flux

$$\boldsymbol{q} = \boldsymbol{q}\hat{} - \tfrac{1}{2}\tau\left[\sum_\alpha m_\alpha\int \boldsymbol{V}_\alpha V_\alpha^2\frac{\partial}{\partial t}(f_\alpha^{(0)}\phi_\alpha)\,\mathrm{d}\boldsymbol{V}_\alpha + m_\alpha\int \boldsymbol{V}_\alpha V_\alpha^2\frac{\partial}{\partial \boldsymbol{r}}\cdot(\boldsymbol{v}_\alpha f_\alpha^{(0)}\phi_\alpha)\,\mathrm{d}\boldsymbol{v}_\alpha\right]. \tag{3.41}$$

The calculations lead to the result:

$$\boldsymbol{q} = \boldsymbol{q}^{\wedge} - \tau \left\{ \frac{\partial \boldsymbol{q}^{\wedge}}{\partial t} - 2\eta \left(\boldsymbol{v}_0 \cdot \left[\frac{\partial}{\partial \boldsymbol{r}} \right) \left(\boldsymbol{v}_0 \cdot \right] \boldsymbol{S} \right) - 2\eta (\boldsymbol{v}_0 \cdot \boldsymbol{S}) \cdot \frac{\partial}{\partial \boldsymbol{r}} \boldsymbol{v}_0 \right.$$
$$+ \frac{\partial}{\partial \boldsymbol{r}} \cdot (\eta \boldsymbol{S} v_0^2) - v_0^2 \frac{\partial}{\partial \boldsymbol{r}} \cdot (\eta \boldsymbol{S}) + 7 \frac{\partial}{\partial \boldsymbol{r}} \cdot \left[\boldsymbol{S} \sum_\alpha \frac{p_\alpha}{\rho_\alpha} (\eta_\alpha^1 - \eta) \right]$$
$$- \check{k} \left[\frac{\partial}{\partial \boldsymbol{r}} \boldsymbol{v}_0 \cdot \frac{\partial T}{\partial \boldsymbol{r}} + \frac{\partial T}{\partial \boldsymbol{r}} \cdot \frac{\partial}{\partial \boldsymbol{r}} \boldsymbol{v}_0 + \frac{\partial T}{\partial \boldsymbol{r}} \frac{\partial}{\partial \boldsymbol{r}} \cdot \boldsymbol{v}_0 \right] + \frac{pn}{\rho} \left[\frac{\partial}{\partial \boldsymbol{r}} \boldsymbol{v}_0 \cdot \sum_{\alpha\beta} m_\beta (D_{\alpha\beta} - D_{\alpha\beta}^1)\, \boldsymbol{d}_\beta \right.$$
$$\left. + \sum_\alpha m_\beta (D_{\alpha\beta} - D_{\alpha\beta}^1)\, \boldsymbol{d}_\beta \cdot \frac{\partial}{\partial \boldsymbol{r}} \boldsymbol{v}_0 + \sum_{\alpha\beta} m_\beta (D_{\alpha\beta} - D_{\alpha\beta}^1)\, \boldsymbol{d}_\beta \frac{\partial}{\partial \boldsymbol{r}} \cdot \boldsymbol{v}_0 \right]$$
$$\left. + \frac{5}{2} \frac{p}{\rho} \sum_{\alpha\beta} m_\beta (D_{\alpha\beta} - D_{\alpha\beta}^1) \left(\boldsymbol{d}_\beta \cdot \frac{\partial}{\partial \boldsymbol{r}} \right) \boldsymbol{v}_0 - \tfrac{5}{2} \check{k} \left(\frac{\partial T}{\partial \boldsymbol{r}} \cdot \frac{\partial}{\partial \boldsymbol{r}} \right) \boldsymbol{v}_0 \right\}. \tag{3.42}$$

From (3.42) we can conclude that the heat flux $\boldsymbol{q}$ depends on viscosity and diffusion in the mixture of gases. We must write down several other integrals for the average values on the right-hand side of (3.42):

$$\frac{\partial^2}{\partial \boldsymbol{r}\, \partial \boldsymbol{r}} : \sum_\alpha \rho_\alpha \overline{(\boldsymbol{V}_\alpha \boldsymbol{V}_\alpha)(\boldsymbol{V}_\alpha \cdot \boldsymbol{v}_0)} = -2 \frac{\partial^2}{\partial \boldsymbol{r}\, \partial \boldsymbol{r}} : \check{k} \boldsymbol{v}_0 \frac{\partial T}{\partial \boldsymbol{r}} - \Delta \left(\check{k} \boldsymbol{v}_0 \cdot \frac{\partial T}{\partial \boldsymbol{r}} \right)$$
$$-2 \frac{\partial^2}{\partial \boldsymbol{r}\, \partial \boldsymbol{r}} : \frac{pn}{\rho} \boldsymbol{v}_0 \sum_{\alpha\beta} m_\alpha m_\beta (D_{\alpha\beta} - D_{\alpha\beta}^1)\, \boldsymbol{d}_\beta - \Delta \left[\frac{pn}{\rho} \boldsymbol{v}_0 \cdot \sum_{\alpha\beta} m_\alpha m_\beta (D_{\alpha\beta} - D_{\alpha\beta}^1)\, \boldsymbol{d}_\beta \right]; \tag{3.43}$$

$$\frac{1}{2} \frac{\partial^2}{\partial \boldsymbol{r}\, \partial \boldsymbol{r}} : \sum_\alpha \rho_\alpha \overline{(\boldsymbol{V}_\alpha \boldsymbol{V}_\alpha) V_\alpha^2} = \tfrac{5}{3} k \Delta \left(T \sum_\alpha \frac{p_\alpha}{m_\alpha} \right) - \frac{56}{9} \frac{\partial^2}{\partial \boldsymbol{r}\, \partial \boldsymbol{r}} : \left[\boldsymbol{S} k T \sum_\alpha \frac{\eta_\alpha - \eta_\alpha^1}{m_\alpha} \right]$$
$$- \frac{140}{27} \frac{\partial^2}{\partial \boldsymbol{r}\, \partial \boldsymbol{r}} : \left[Д k T \sum_\alpha \frac{\eta_\alpha - \eta_\alpha^1}{m_\alpha} \right], \tag{3.44}$$

where the tensor Д has the components

$$Д_{ij} = \delta_{ij}\, \partial v_{0i} / \partial r_j, \quad i, j = 1, 2, 3 \tag{3.45}$$

and Δ is laplacian. At last,

$$\frac{\partial^2}{\partial \boldsymbol{r}\, \partial \boldsymbol{r}} : \sum_\alpha \rho_\alpha \overline{(\boldsymbol{V}_\alpha \boldsymbol{v}_0)\, \boldsymbol{V}_\alpha \cdot \boldsymbol{v}_0} = \frac{\partial^2}{\partial \boldsymbol{r}\, \partial \boldsymbol{r}} : p \boldsymbol{v}_0 \boldsymbol{v}_0 - 2 \frac{\partial^2}{\partial \boldsymbol{r}\, \partial \boldsymbol{r}} : \eta (\boldsymbol{v}_0 \boldsymbol{v}_0 \cdot \boldsymbol{S}), \tag{3.46}$$

$$\frac{\partial^2}{\partial \boldsymbol{r}\, \partial \boldsymbol{r}} : \sum_\alpha \rho_\alpha \overline{(\boldsymbol{V}_\alpha \boldsymbol{v}_0) V_\alpha^2} = -5 \frac{\partial^2}{\partial \boldsymbol{r}\, \partial \boldsymbol{r}} : \check{k} \frac{\partial T}{\partial \boldsymbol{r}} \boldsymbol{v}_0 + 5 \frac{\partial^2}{\partial \boldsymbol{r}\, \partial \boldsymbol{r}} : \frac{p}{\rho} \sum_{\alpha\beta} m_\beta (D_{\alpha\beta} - D_{\alpha\beta}^1)\, \boldsymbol{d}_\beta \boldsymbol{v}_0. \tag{3.47}$$

Several concrete examples based on these generalized hydrodynamical equations will be considered in the following sections.

(c) *Strict turbulence theory*

During many years' intensive research activity in the field of turbulent flows, applications associated with jet and rocket propulsion, material processing industries and so on have been developed. The state of the research, classification of various models and their applications and limitations (see, for example, Kuo 1986) will not be discussed. I restrict discussion of modern turbulence theory to several remarks that are of principal importance to the following theory.

1. The turbulent motion is random and irregular; therefore there is a broad range of length scales.

2. Turbulent flows are always dissipative, and the smallest scale (Kolmogorov scale) in turbulence is connected with molecular viscosity.

3. Turbulence originates as an instability of laminar flows at high Reynolds numbers.

4. Turbulence is governed by the same equations (in the definite sense) of fluid mechanics as laminar flow.

The last affirmation must be discussed in detail. The first step of a turbulence model construction consists in averaging Navier–Stokes equations with the help of a time-averaging procedure (Reynolds averaging) or mass-weighted averaging (Favre averaging). Other methods for the fluid mechanical description of the problem have their shortcomings; for example, application of the moment methods is connected inevitably with the unclosed system of moments. The origin of all limitations of known turbulence models is their phenomenological character. There are two implicit suggestions in the classical turbulence theory: (i) Navier–Stokes equations are adequate for turbulence description; (ii) fluctuation terms are proportional to any previously unknown turbulent viscosity. As a result the governing equations take unclosed form, because there are more unknowns than equations. On the other hand, at least on the Kolmogorov scale the base equations of fluid mechanics must contain fluctuation terms in explicit form, because this smallest scale in a turbulent flow is a dissipative microscale, connected with usual viscosity. In this sense Navier–Stokes equations do not contain turbulence, and effects of turbulence must be introduced in the classical theory in a phenomenological way, like the Reynolds procedure. But Navier–Stokes equations are the direct consequence of the Boltzmann equation. Therefore the new theory of turbulence must be based on the other kinetic equations. From this point of view the generalized hydrodynamic equations have a very interesting structure. The additional terms in the GHE are proportional to viscosity and correspond to fluctuation terms on the Kolmogorov scale $l_K = (\nu^3/\epsilon_d)^{\frac{1}{4}}$, where ν is the kinematic viscosity and ϵ_d is the rate of energy dissipation per unit fluid mass. Therefore these equations should be discussed from turbulence positions. But for this purpose we must answer two main questions. (1) Is it possible to organize the explicit form, on the level of the generalized Enskog hydrodynamic equations, of the Reynolds procedure of extracting from the 'real' hydrodynamic values the fluctuation terms? (2) Is it possible to prove the absence of contradictions in the obtained fluctuations? The answers are both yes. First of all the GHE must be rewritten in the divergent form, as it was done with the continuity equation (3.10). For example, the momentum equation (assuming that the magnetic field is absent) can be transformed as

$$\begin{aligned}
&\frac{\partial}{\partial t}\left\{\rho \boldsymbol{v}_0-\tau\left[\frac{\partial}{\partial t}(\rho \boldsymbol{v}_0)+\frac{\partial}{\partial \boldsymbol{r}}\cdot(\boldsymbol{P}+\rho \boldsymbol{v}_0\,\boldsymbol{v}_0)-\sum_\alpha \rho_\alpha \boldsymbol{F}_\alpha^{(1)}\right]\right\} \\
&\quad-\left\{\sum_\alpha \rho_\alpha \boldsymbol{F}_\alpha^{(1)}-\tau\left[\sum_\alpha \boldsymbol{F}_\alpha^{(1)}\frac{\partial \rho_\alpha}{\partial t}+\sum_\alpha \boldsymbol{F}_\alpha^{(1)}\frac{\partial}{\partial \boldsymbol{r}}\cdot(\rho_\alpha \overline{\boldsymbol{v}}_\alpha)\right]\right\} \\
&\quad+\frac{\partial}{\partial \boldsymbol{r}}\cdot\left\{\boldsymbol{P}+\rho \boldsymbol{v}_0\,\boldsymbol{v}_0-\tau\left[\frac{\partial}{\partial t}(\boldsymbol{P}+\rho \boldsymbol{v}_0\,\boldsymbol{v}_0)+\frac{\partial}{\partial \boldsymbol{r}}\cdot[\boldsymbol{P}\boldsymbol{v}_0+(\rho \boldsymbol{v}_0\,\boldsymbol{v}_0)\,\boldsymbol{v}_0+\sum_\alpha \rho_\alpha\overline{(\boldsymbol{V}_\alpha \boldsymbol{v}_0)\,\boldsymbol{V}_\alpha}\right.\right. \\
&\quad\left.\left.+\sum_\alpha \rho_\alpha\overline{(\boldsymbol{v}_0\,\boldsymbol{V}_\alpha)\,\boldsymbol{V}_\alpha}+\sum_\alpha \rho_\alpha\overline{(\boldsymbol{V}_\alpha\,\boldsymbol{V}_\alpha)\,\boldsymbol{V}_\alpha}]-\sum_\alpha \boldsymbol{F}_\alpha^{(1)}\rho_\alpha \overline{\boldsymbol{v}}_\alpha-\sum_\alpha \rho_\alpha \overline{\boldsymbol{v}}_\alpha \boldsymbol{F}_\alpha^{(1)}\right]\right\}=0. \qquad (3.48)
\end{aligned}$$

The energy equation in the divergence form is

$$\frac{\partial}{\partial t}\Big\{\tfrac{1}{2}\rho v_0^2+\tfrac{3}{2}p+\sum_\alpha \epsilon_\alpha n_\alpha-\tau\Big[\frac{\partial}{\partial t}(\tfrac{1}{2}\rho v_0^2+\tfrac{3}{2}p+\sum_\alpha \epsilon_\alpha n_\alpha)$$

$$+\frac{\partial}{\partial \boldsymbol{r}}\cdot[\boldsymbol{v}_0\cdot(\boldsymbol{P}+\boldsymbol{I}(\tfrac{1}{2}\rho v_0^2+\tfrac{3}{2}p+\sum_\alpha \epsilon_\alpha n_\alpha))+\boldsymbol{q}]-\sum_\alpha \boldsymbol{F}_\alpha^{(1)}\cdot\rho_\alpha\overline{\boldsymbol{v}}_\alpha\Big]\Big\}$$

$$+\frac{\partial}{\partial \boldsymbol{r}}\cdot\Big\{\boldsymbol{v}_0(\tfrac{1}{2}\rho v_0^2+\tfrac{3}{2}p)+\boldsymbol{v}_0\cdot\boldsymbol{P}+\boldsymbol{v}_0\sum_\alpha \epsilon_\alpha n_\alpha+\boldsymbol{q}-\tau\Big[\frac{\partial}{\partial t}(\boldsymbol{v}_0(\tfrac{1}{2}\rho v_0^2+\tfrac{3}{2}p+\sum_\alpha \epsilon_\alpha n_\alpha)$$

$$+\boldsymbol{v}_0\cdot\boldsymbol{P}+\boldsymbol{q})+\frac{\partial}{\partial \boldsymbol{r}}\cdot\Big[\boldsymbol{v}_0\boldsymbol{v}_0\tfrac{1}{2}\rho v_0^2+2(\boldsymbol{v}_0\cdot\boldsymbol{P})\boldsymbol{v}_0+\tfrac{1}{2}v_0^2\boldsymbol{P}+\sum_\alpha\overline{(\boldsymbol{v}_0\cdot\boldsymbol{V}_\alpha)\rho_\alpha\boldsymbol{V}_\alpha\boldsymbol{V}_\alpha}$$

$$+\tfrac{1}{2}\sum_\alpha\rho_\alpha\overline{\boldsymbol{V}_\alpha\boldsymbol{V}_\alpha V_\alpha^2}+\tfrac{3}{2}p\boldsymbol{v}_0\boldsymbol{v}_0+2\boldsymbol{v}_0\boldsymbol{q}+\sum_\alpha\frac{\epsilon_\alpha}{m_\alpha}(\rho_\alpha\boldsymbol{v}_0\boldsymbol{v}_0+\boldsymbol{P}_\alpha)\Big]$$

$$-\sum_\alpha \boldsymbol{F}_\alpha^{(1)}(\tfrac{1}{2}\rho_\alpha v_0^2+\boldsymbol{j}_\alpha\cdot\boldsymbol{v}_0+\tfrac{3}{2}p_\alpha+\epsilon_\alpha n_\alpha)-\sum_\alpha \boldsymbol{F}_\alpha^{(1)}\cdot(\rho_\alpha\boldsymbol{v}_0\boldsymbol{v}_0+\boldsymbol{v}_0\boldsymbol{j}_\alpha+\boldsymbol{j}_\alpha\cdot\boldsymbol{v}_0+\boldsymbol{P}_\alpha)\Big]\Big\}$$

$$-\Big\{\sum_\alpha \boldsymbol{F}_\alpha^{(1)}\cdot\rho_\alpha\overline{\boldsymbol{v}}_\alpha-\tau\Big[\sum_\alpha \boldsymbol{F}_\alpha^{(1)}\cdot\Big(\frac{\partial}{\partial t}(\rho_\alpha\overline{\boldsymbol{v}}_\alpha)-\rho_\alpha\boldsymbol{F}_\alpha^{(1)}+\frac{\partial}{\partial \boldsymbol{r}}\cdot(\rho_\alpha\boldsymbol{v}_0\boldsymbol{v}_0+\boldsymbol{v}_0\boldsymbol{j}_\alpha+\boldsymbol{j}_\alpha\boldsymbol{v}_0+\boldsymbol{P}_\alpha)\Big)\Big]\Big\}=0. \tag{3.49}$$

In the common case the GHE have the very complex structure. But in many particular cases very good results can be achieved by using only the generalized Euler equations obtained with the help of the local maxwellian function. The origin of this fact is quite obvious; the generalized Navier–Stokes equations contain the fluctuation terms proportional to τ^2 and therefore the generalized Euler equations become a good approximation. The order of the GHE is higher than in classical hydrodynamics. But generally speaking GHE belong to the Cauchy–Kowalewskaya type of equations; this fact leads to the existence and uniqueness of its solution. The problem of additional boundary conditions can be solved if we consider the common symbolic dimensionless form of the GHE:

$$\Phi_{1i}+Kn\,\Phi_{2i}=0, \tag{3.50}$$

where subscript i corresponds to the number of the equation in the system of the GHE. In the surface vicinity where $Kn\rightarrow\infty$, we obtain from (3.50) the additional boundary conditions

$$\Phi_{2i,w}=0. \tag{3.51}$$

Let us take down the system of the GHE in the generalized Euler approximation and consider, using this simpler example, the question of self-consistency of fluctuation terms in the gravitational field. We have the continuity equation

$$\frac{\partial}{\partial t}\Big\{\rho-\tau\Big[\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial \boldsymbol{r}}\cdot(\rho\boldsymbol{v}_0)\Big]\Big\}+\frac{\partial}{\partial \boldsymbol{r}}\cdot\Big\{\rho\boldsymbol{v}_0-\tau\Big[\frac{\partial}{\partial t}(\rho\boldsymbol{v}_0)+\frac{\partial}{\partial \boldsymbol{r}}\cdot(\rho\boldsymbol{v}_0\boldsymbol{v}_0)+\frac{\partial p}{\partial \boldsymbol{r}}-\rho\boldsymbol{g}\Big]\Big\}=0, \tag{3.52}$$

Table 1. *Fluctuations* $[A]^{\mathrm{f}}$ *of hydrodynamic values* $[A]$ *on the Kolmogorov turbulence scale*

hydrodynamic value, $[A]$	fluctuations, $[A]^{\mathrm{f}}$
ρ	$\tau\left(\dfrac{\partial \rho}{\partial t}+\dfrac{\partial}{\partial \boldsymbol{r}}\cdot(\rho \boldsymbol{v}_0)\right)$
$\rho \boldsymbol{v}_0$	$\tau\left[\dfrac{\partial}{\partial t}(\rho \boldsymbol{v}_0)+\dfrac{\partial}{\partial \boldsymbol{r}}\cdot(\rho \boldsymbol{v}_0 \boldsymbol{v}_0)+\dfrac{\partial p}{\partial \boldsymbol{r}}-\rho \boldsymbol{g}\right]$
$p\delta_{\alpha\beta}+\rho v_{0\alpha} v_{0\beta}$	$\tau\left[\dfrac{\partial}{\partial t}(p\delta_{\alpha\beta}+\rho v_{0\alpha} v_{0\beta})+\dfrac{\partial}{\partial r_\gamma}(pv_{0\alpha}\delta_{\beta\gamma}+pv_{0\beta}\delta_{\alpha\gamma}+pv_{0\gamma}\delta_{\alpha\beta}+\rho v_{0\alpha} v_{0\beta} v_{0\gamma})-g_\beta \rho v_{0\alpha}-g_\alpha \rho v_{0\beta}\right]$
$3p+\rho v_0^2$	$\tau\left[\dfrac{\partial}{\partial t}(3p+\rho v_0^2)+\dfrac{\partial}{\partial \boldsymbol{r}}\cdot(\boldsymbol{v}_0(\rho v_0^2+5p))-2\boldsymbol{g}\cdot\rho \boldsymbol{v}_0\right]$
$\boldsymbol{v}_0(\rho v_0^2+5p)$	$\tau\left\{\dfrac{\partial}{\partial t}(\boldsymbol{v}_0(\rho v_0^2+5p))+\dfrac{\partial}{\partial \boldsymbol{r}}\cdot\left(\rho v_0^2 \boldsymbol{v}_0\boldsymbol{v}_0+\boldsymbol{I}pv_0^2+7p\boldsymbol{v}_0 \boldsymbol{v}_0+5\boldsymbol{I}\dfrac{p^2}{\rho}\right)-2\rho \boldsymbol{v}_0 \boldsymbol{v}_0\cdot \boldsymbol{g} -5p\boldsymbol{I}\cdot\boldsymbol{g}-\rho v_0^2\boldsymbol{I}\cdot\boldsymbol{g}-p\left[\dfrac{\partial}{\partial \boldsymbol{r}}\left(5\dfrac{p}{\rho}+v_0^2\right)+2\left(\boldsymbol{v}_0\cdot\dfrac{\partial}{\partial \boldsymbol{r}}\right)\boldsymbol{v}_0-\tfrac{4}{3}\boldsymbol{v}_0\left(\dfrac{\partial}{\partial \boldsymbol{r}}\cdot\boldsymbol{v}_0\right)\right]\right\}$

the momentum equation in the coordinate form (α, β, $\gamma=1,2,3$)

$$\frac{\partial}{\partial t}\left\{\rho v_{0\beta}-\tau\left[\frac{\partial}{\partial t}(\rho v_{0\beta})+\frac{\partial}{\partial r_\alpha}(p\delta_{\alpha\beta}+\rho v_{0\alpha})-\rho g_\beta\right]\right\}$$
$$-\left\{\rho-\tau\left[\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial r_\alpha}(\rho v_{0\alpha})\right]\right\}g_\beta+\frac{\partial}{\partial r_\alpha}\left\{p\delta_{\alpha\beta}+\rho v_{0\alpha} v_{0\beta}-\tau\left[\frac{\partial}{\partial t}(p\delta_{\alpha\beta}+\rho v_{0\alpha} v_{0\beta})\right.\right.$$
$$\left.\left.+\frac{\partial}{\partial r_\gamma}(pv_{0\beta}\delta_{\alpha\gamma}+pv_{0\alpha}\delta_{\beta\gamma}+pv_{0\gamma}\delta_{\alpha\beta}+\rho v_{0\alpha} v_{0\beta} v_{0\gamma})-g_\alpha \rho v_{0\beta}-\rho v_{0\alpha} g_\beta\right]\right\}=0, \tag{3.53}$$

the energy equation

$$\frac{\partial}{\partial t}\left\{3p+\rho v_0^2-\tau\left[\frac{\partial}{\partial t}(3p+\rho v_0^2)+\frac{\partial}{\partial \boldsymbol{r}}\cdot(\boldsymbol{v}_0(\rho v_0^2+5p))-2\boldsymbol{g}\cdot\rho \boldsymbol{v}_0\right]\right\}$$
$$+\frac{\partial}{\partial \boldsymbol{r}}\cdot\left\{\boldsymbol{v}_0(\rho v_0^2+5p)-\tau\left[\frac{\partial}{\partial t}(\boldsymbol{v}_0(\rho v_0^2+5p))+\frac{\partial}{\partial \boldsymbol{r}}\cdot\left[\boldsymbol{I}pv_0^2+\rho v_0^2 \boldsymbol{v}_0 \boldsymbol{v}_0+7p\boldsymbol{v}_0 \boldsymbol{v}_0+5\boldsymbol{I}\frac{p^2}{\rho}\right]\right.\right.$$
$$\left.\left.-2\rho \boldsymbol{v}_0 \boldsymbol{v}_0\cdot g-5p\boldsymbol{I}\cdot\boldsymbol{g}-\rho v_0^2\boldsymbol{I}\cdot\boldsymbol{g}\right]\right\}-2\boldsymbol{g}\cdot\left\{\rho \boldsymbol{v}_0-\tau\left[\frac{\partial}{\partial t}(\rho \boldsymbol{v}_0)+\frac{\partial p}{\partial \boldsymbol{r}}+\frac{\partial}{\partial \boldsymbol{r}}\cdot(\rho \boldsymbol{v}_0 \boldsymbol{v}_0)-\rho \boldsymbol{g}\right]\right\}=0. \tag{3.54}$$

Therefore we have for the case considered the following system of fluctuations (for hard spheres $\tau=0.786\eta p^{-1}$) indicated in table 1.

Obviously, there are no contradictions in the system of fluctuations in table 1. We can prove for example

$$\sum_{\substack{\alpha,\beta\\ \alpha=\beta}}(p\delta_{\alpha\beta}+\rho v_{0\alpha} v_{0\beta})^{\mathrm{f}}=(3p+\rho v_0^2)^{\mathrm{f}}. \tag{3.55}$$

Such procedures can be realized in the common case for the generalized hydrodynamic Enskog equations. Table 1 could be extended to include other fluctuations. For example, the product of 'real' density and 'real' velocity $\rho \boldsymbol{v}_0$ can be transformed to get the fluctuation value $\boldsymbol{v}_0^{\mathrm{f}}$. Really, $\rho \boldsymbol{v}_0 = (\rho^{\mathrm{a}} + \rho^{\mathrm{f}})(\boldsymbol{v}_0^{\mathrm{a}} + \boldsymbol{v}_0^{\mathrm{f}})$ and omitting the small squared fluctuation terms we have

$$\boldsymbol{v}_0^{\mathrm{f}} = \frac{1}{\rho}[(\rho \boldsymbol{v}_0)^{\mathrm{f}} - \rho^{\mathrm{f}} \boldsymbol{v}_0] \tag{3.56}$$

or

$$\boldsymbol{v}_0^{\mathrm{f}} = \tau \left[\frac{\partial \boldsymbol{v}_0}{\partial t} + \left(\boldsymbol{v}_0 \cdot \frac{\partial}{\partial \boldsymbol{r}} \right) \boldsymbol{v}_0 + \frac{1}{\rho} \frac{\partial p}{\partial \boldsymbol{r}} - \boldsymbol{g} \right]. \tag{3.57}$$

By analogy,

$$p^{\mathrm{f}} = \tau \left[\frac{\partial p}{\partial t} + \frac{\partial}{\partial \boldsymbol{r}} \cdot (p \boldsymbol{v}_0) + \tfrac{2}{3} p \frac{\partial}{\partial \boldsymbol{r}} \cdot \boldsymbol{v}_0 \right]. \tag{3.58}$$

Let us take down the generalized Euler equations using the terminology of 'averaged' values:

$$\frac{\partial \rho^{\mathrm{a}}}{\partial t} + \frac{\partial}{\partial \boldsymbol{r}} \cdot (\rho \boldsymbol{v}_0)^{\mathrm{a}} = 0, \tag{3.59}$$

$$\frac{\partial}{\partial t} (\rho \boldsymbol{v}_0)^{\mathrm{a}} + \frac{\partial}{\partial \boldsymbol{r}} \cdot [p^{\mathrm{a}} \boldsymbol{I} + (\rho \boldsymbol{v}_0 \boldsymbol{v}_0)^{\mathrm{a}}] = \rho^{\mathrm{a}} \boldsymbol{g}, \tag{3.60}$$

$$\frac{\partial}{\partial t}(3p + \rho v_0^2)^{\mathrm{a}} + \frac{\partial}{\partial \boldsymbol{r}} \cdot [\boldsymbol{v}_0(\rho v_0^2 + 5p)]^{\mathrm{a}} - 2\boldsymbol{g} \cdot (\rho \boldsymbol{v}_0)^{\mathrm{a}}$$
$$= \frac{\partial}{\partial \boldsymbol{r}} \cdot \left\{ \tau p \left[\frac{\partial}{\partial \boldsymbol{r}} \left(5 \frac{p}{\rho} + v_0^2 \right) + 2 \left(\boldsymbol{v}_0 \cdot \frac{\partial}{\partial \boldsymbol{r}} \right) \boldsymbol{v}_0 - \tfrac{4}{3} \boldsymbol{v}_0 \left(\frac{\partial}{\partial \boldsymbol{r}} \cdot \boldsymbol{v}_0 \right) \right] \right\}. \tag{3.61}$$

We must remember, that τp is proportional to the viscosity η. The system (3.59)–(3.61) has a very interesting structure. If the dissipative term in the right-hand side of the energy equation (3.61) is small (this term is connected only with 'real' hydrodynamical values) and could be omitted, (3.59)–(3.61) are the usual Euler equations written for mean hydrodynamical values. Then we can state that the terms in the energy equation

$$\frac{\partial}{\partial \boldsymbol{r}} \cdot \left\{ \eta \left[\frac{5}{2} \frac{\partial}{\partial \boldsymbol{r}} \left(\frac{p}{\rho} \right) + 2 \boldsymbol{v}_0 \cdot \boldsymbol{S} \right] \right\}$$

provoke the turbulence in the physical system considered.

4. Application of the generalized hydrodynamical equations

(*a*) *Propagation of forced sound waves in rarefied gas dynamics*

The problem of forced sound waves propagation is the classical subject of investigation in rarefied gas dynamics. Let us consider a flat plate with infinite dimensions oscillating with a frequency ω in an infinite volume of gas. These oscillations generate sound waves propagating in the normal (to the surface of the plate) direction. We introduce the parameter $a = \omega \tau$, where τ is the free mean time between collisions. For the hard sphere model in rarefied gas the relation $p\tau = \Pi \mu$ is valid ($\Pi = 0.786$). The parameter a can be connected analogously with the Reynolds number $r = \Pi / a$. For large r the classical gas dynamics lead to quite satisfactory results. In linear acoustics where r tends to infinity the damping of sounds tends to zero and the sound speed can be received as $c_0^2 = \gamma p_0 / \rho_0$, where ρ_0 is the density of unperturbed gas, $\gamma = cp/c_V$ is the ratio of constant pressure specified heat to volume constant specific heat. For $r \sim 1$, moreover for $r \to 0$, the situation becomes very complicated. Euler equations do not observe the changed situation and these equations lead to constant sound speed and zero damping for all the diapason of r. Navier–Stokes equations also lead to unsatisfactory result: after achieving a maximum by $r \sim 1$ the damping of sound tends to zero and sound speed tends to infinity when $r \to 0$. Therefore for small r the investigation is building on the kinetic level of the description. Concerning the methods of physical kinetics, particularly moment methods (see, for example, Cercignani 1988) we should take note of the whole unsatisfactory picture. For example, using moment methods leads to a change for the worse if the number of moments increases. The numerical results known were obtained with the help of hundreds of moments (Cercignani 1988).

Let us consider now the possibilities of the generalized hydrodynamical equations in calculating the speed of sound and the damping of sound for arbitrary r-numbers. In 'linear' acoustics the perturbation of density and temperature can be written in the form:

$$\rho = \rho_0(1+s_\rho), \quad s_\rho \sim \exp(\mathrm{i}\omega t - kx), \tag{4.1}$$

$$T = T_0(1+s_T), \quad s_T \sim \exp(\mathrm{i}\omega t - kx), \tag{4.2}$$

where k, generally speaking, is a complex number. Linearizing the GHE and using the relations (4.1) and (4.2) we find the following dispersion algebraic equations originating from the continuity, momentum and energy equations in the generalized Navier–Stokes approximation:

$$\begin{aligned}
&\{1008AB-180A\}\frac{\kappa^5}{\Pi^2}a^7\hat{k}^{10}+\bigg\{-\frac{1008}{\Pi^2}a^7AB+\frac{156}{\Pi^2}a^7A+\frac{672}{\Pi^2}a^7B-\frac{210}{\Pi^2}a^7\\
&\quad-\frac{66}{\Pi^2}a^7A-\frac{168}{\Pi^2}a^5B+\frac{40}{\Pi}a^5A+\frac{378}{\Pi}a^5B+\frac{90}{\Pi^2}a^5-\frac{135}{2\Pi}a^5\\
&\quad+\mathrm{i}\frac{a^6}{\Pi^2}[-1008AB+222A+672B-270]\bigg\}\kappa^4\hat{k}^8+\bigg\{-\frac{384}{\Pi^2}a^7A+\frac{336}{\Pi^2}a^7B+\frac{46}{\Pi^2}a^7\\
&\quad+\frac{726}{\Pi^2}a^5A-\frac{672}{\Pi^2}a^5B-\frac{80}{\Pi}a^5A-\frac{36}{\Pi^2}a^5-\frac{137}{6\Pi}a^5-\frac{10}{\Pi^2}a^3+\frac{95}{6\Pi}a^3+15a^3-\frac{126}{\Pi}a^3B\\
&\quad+\mathrm{i}\bigg[-\frac{918}{\Pi^2}a^6A+\frac{840}{\Pi^2}a^6B+\frac{88}{\Pi^2}a^6+\frac{192}{\Pi^2}a^4A-\frac{168}{\Pi^2}a^4B-\frac{80}{\Pi}a^4A\\
&\quad+\frac{36}{\Pi^2}a^4-\frac{527}{6\Pi}a^4\bigg]\bigg\}\kappa^3\hat{k}^6+\bigg\{-\frac{40}{\Pi^2}a^7-\frac{152}{3\Pi}a^5A+\frac{126}{\Pi}a^5B+\frac{250}{\Pi^2}a^5-\frac{245}{6\Pi}a^5\\
&\quad+\frac{124}{3\Pi}a^3A-\frac{126}{\Pi}a^3B+\frac{337}{6\Pi}a^3-\frac{70}{\Pi^2}a^3-15a^3-\frac{15}{2\Pi}a-5a\\
&\quad+\mathrm{i}\bigg[-\frac{160}{\Pi^2}a^6-\frac{120}{\Pi}a^4A+\frac{252}{\Pi}a^4B+\frac{190}{\Pi^2}a^4-\frac{232}{3\Pi}a^4+\frac{103}{6\Pi}a^2-\frac{28}{\Pi}a^2A-15a^2\bigg]\bigg\}\kappa^2\hat{k}^4\\
&\quad+\bigg\{-\frac{23}{\Pi}a^5+\frac{207}{2\Pi}a^3-a^3-\frac{23}{2\Pi}a+6a+\mathrm{i}\bigg[-\frac{161}{2\Pi}a^4+\frac{115}{2\Pi}a^2-2a^2+5\bigg]\bigg\}\kappa\hat{k}^2\\
&\quad+\{-3a^3+9a+\mathrm{i}[-9a^2+3]\}=0,
\end{aligned}\tag{4.3}$$

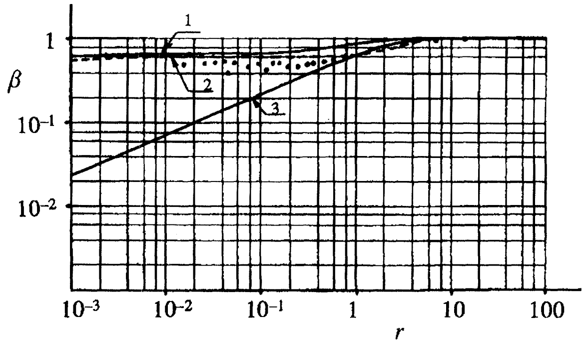

Figure 1. Forced sound-wave speeds: 1, generalized Euler equation; 2, generalized Navier–Stokes equation; 3, Navier-Stokes equation; ○, ●, comparison with experiments (Greenspan 1956; Meyer & Sessler 1957 respectively).

where $\kappa = \gamma^{-1}$, $A = 1-\eta^1/\eta$, $B = 1-\lambda^{1\prime}/(2\lambda')$; $\eta^1, \lambda^{1\prime}$ are additional coefficients of viscosity and heat conduction, which (as distinct from the usual viscosity η and heat conduction λ') must be calculated using the first and second coefficients respectively in the Sonine expansion. For example, with $\eta^1 = 0.5pb_1$, the next relation is valid for the dimensionless wave number

$$\hat{k} = \frac{kc_0}{\omega} = \alpha + \mathrm{i}\beta. \tag{4.4}$$

The dispersional equation (4.3) has two obvious asymptotics:

(*a*) $r \to \infty,\ (a \to 0), 5\kappa\hat{k}^2 + 3 = 0$ (4.5)

(*b*) $r \to 0, (a \to \infty), (336AB - 60A)\,\kappa^3\hat{k}^6$

$$+(-336AB + 52A + 224B - 70)\,\kappa^2\hat{k}^4 + [-128A + 112B + \tfrac{46}{3}]\,\kappa\hat{k}^2 - \tfrac{40}{3} = 0. \tag{4.6}$$

For the hard spheres model: $A = 0.9415$; $B = 0.956$. The results of calculations for case (*b*) are $\alpha = 0.426$; $\beta = 0.567$. The dispersion equation for the generalized Euler model can be expressed as an algebraic equation of the sixth degree:

$$3\kappa^2a^3\hat{k}^6 - \kappa a(1+3a^2+3a\mathrm{i})\,\hat{k}^4 + [\tfrac{1}{5}a(6-a^2) + \mathrm{i}(1-\tfrac{2}{5}a^2)]\,\hat{k}^2 + \frac{3}{5\kappa}[3a - a^3 + \mathrm{i}(1-3a^2)] = 0. \tag{4.7}$$

In the limit case of low frequencies ($a \to 0$, continuum range) (4.7) has the 'physical' root $\hat{k} = \mathrm{i}$, therefore $\alpha = 0$, $c = c_0$. In the limit case of high frequencies, $a \to \infty$. Equation (4.7) has the solution $k = 0.509 + \mathrm{i}0.650$.

Figure 1 contains the results for the dimensionless speed of sound $\beta = c/c_0$ for five theoretical models in the case of hard spheres and experimental values (Greenspan 1956; Meyer & Sessler 1957). In figure 2 the corresponding values for the damping of sound α are plotted (the results of the hydrodynamical models, existing experimental data and the results of moment methods).

Several remarks and conclusions follow.

1. In addition to employing monoatomic argon, Meyer & Sessler also used more complicated gases (H_2, air). But in the Knudsen region strong agreement exists between all experimental results because at relatively high frequencies the internal degrees of freedom do not have an opportunity to be excited. The results, received

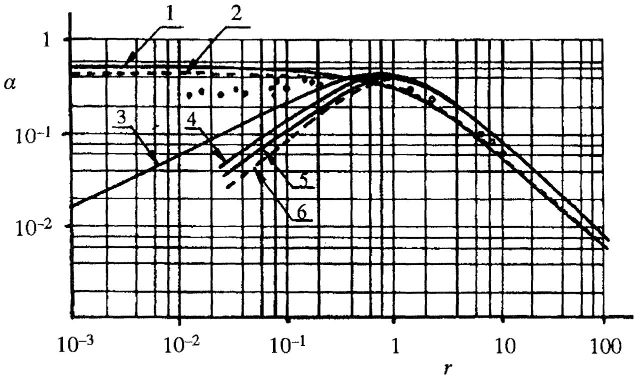

Figure 2. Forced sound-wave attenuation rates: 1, generalized Euler equation; 2, generalized Navier–Stokes equation; 3, Navier–Stokes equation; 4, Burnett equation; 5, Super-Burnett equation, 6, 105 moment equation.

with the help of the GHE, extend well past existing experimental values; the generalized Navier–Stokes curve is in better agreement with experimental data than generalized Euler curve.

2. It is a significant fact that using formal more accurate kinetic models (the results obtained by 105 and 483 moment truncation) leads to worse results. And also using formal more accurate hydrodynamical models (Navier–Stokes, Burnett, Super-Burnett) leads to worse results. Therefore, the 'better' the model the worse the result. However all these models are based on the classical Boltzmann equation. Therefore only the generalized Boltzmann equation can lead to correction of results for the intermediate Knudsen number region.

3. The question arises as to why GHE are so effective in the intermediate Knudsen number region, where it seems GHE would fail. It is connected with the 'interpolative' structure of these equations. Let us consider the GBE:

$$\frac{\mathscr{D}\hat{f}}{\mathscr{D}\hat{t}} - Kn\frac{\mathscr{D}}{\mathscr{D}\hat{t}}\frac{\mathscr{D}\hat{f}}{\mathscr{D}\hat{t}} = \frac{1}{Kn}\hat{J}. \tag{4.8}$$

If Kn tends to infinity (4.8) leads to the result

$$\frac{\mathscr{D}}{\mathscr{D}\hat{t}}\frac{\mathscr{D}\hat{f}}{\mathscr{D}\hat{t}} = 0 \tag{4.9}$$

and therefore the equation $\mathscr{D}\hat{f}/\mathscr{D}\hat{t} = 0$ is the usual limit case well known in rarefied gas dynamics. From this point of view the second term on the left-hand side of (4.9) is realizing 'interpolation' between the continual limit and the free molecular régime limit. Of course the theory of GBE and GHE carries out the first steps in applications only, and the range of possibilities of the above theory will clear up only in the future. And now we consider the last example: the structure of shock wave in monoatomic gas.

(b) *Shock wave structure*

The problem of shock wave structure has been tackled theoretically by many authors both in terms of the Navier–Stokes equations as well as by kinetic theory approaches. I have not observed very many results obtained in calculations of the shock wave structure (see, for example, Alexeev & Ustjugov 1988; Alexeev & Polev

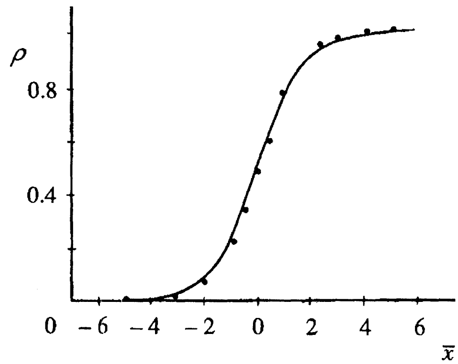

Figure 3. Dimensionless density $\bar{\rho}$ variation in shock wave. $M = 4$; ●, experimental results (Schmidt 1969).

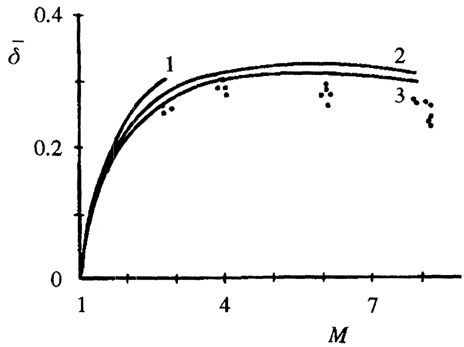

Figure 4. Dimensionless thickness $\bar{\delta}$ variation: 1, Navier–Stokes equation; 2, generalized Euler equation; 3, generalized Navier–Stokes equation; ●, experimental results (Schmidt 1969).

1991). I must emphasize, though, that the application of the Navier–Stokes equations for investigation of shock wave structure in monoatomic gas leads to unacceptable results if the Mach number exceeds 1.6.

Let us consider the structure of a plane stationary shock wave in monoatomic gas with the help of the GHE. According to the physical concept formulated above, GHE, for example in the generalized Euler approximation, can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}x}(\rho v_0) - \tau \frac{\mathrm{d}^2}{\mathrm{d}x^2}(p + \rho v_0^2) = 0, \tag{4.10}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(p + \rho v_0^2) - \tau \frac{\mathrm{d}^2}{\mathrm{d}x^2}(3v_0 p + \rho v_0^3) = 0, \tag{4.11}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[v_0(\rho v_0^2 + 5p)] - \tau \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\rho v_0^4 + 8p v_0^2 + 5\frac{p^2}{\rho}\right) = 0. \tag{4.12}$$

The system (4.10)–(4.12) can be easily integrated using the iterative procedure. As usual the boundary conditions for (4.10)–(4.12) are Rankine–Hugoniot relations. Figures 3 and 4 show a sample calculation for argon and the comparison of calculated values with experimental data (Schmidt 1969). Figure 3 contains the dimensionless density variation ($\bar{\rho} = (\rho - \rho_1)/(\rho_2 - \rho_1)$, where ρ_1, ρ_2 are the densities in upstream and downstream gas) for $M = 4$ with respect to dimensionless length

$$\bar{x} = \frac{x}{x_1}, \quad x_1 = \frac{16}{5}\sqrt{\left(\frac{5}{6\pi}\right)}\frac{\eta_1}{\rho_1 c_1},$$

where c_1 is the sound speed in the upstream region. Experimental values are shown as dots. Figure 4 shows the dimensionless thickness $\bar{\delta}$ of shock wave for different Mach numbers,

$$\bar{\delta} = \frac{l}{d}, \quad d = \frac{\rho_2 - \rho_1}{(\mathrm{d}\rho/\mathrm{d}x)_{\max}}.$$

In figure 4 the curves 1, 2, 3 correspond to Navier–Stokes, generalized Euler and generalized Navier–Stokes solutions respectively. All calculations received in the frame of GHE are in good agreement with experimental data.

5. Concluding remarks

1. The generalized Boltzmann equation takes into account the alteration of the distribution function on scales of collision time. The fact of great importance is that additional terms are proportional to the Knudsen number and therefore, in the hydrodynamical limit, these terms are proportional to viscosity.

2. The generalized Boltzmann equation can be considered an analogy of the Fokker–Plank equation defining the diffusion in the phase-space. The generalized Boltzmann equation inevitably leads to the new hydrodynamical theory and the classical Navier–Stokes and Euler equations are only a particular case of this theory.

3. The generalized hydrodynamical equations can be considered as the base of the strict theory of turbulence on the Kolmogorov scale of turbulence. Additional boundary conditions for these equations coincide with the well known conditions in the turbulence theory connected with the disappearance of fluctuations on the wall. The generalized hydrodynamical equations have, by force of their structure, the very effective 'interpolative' properties for the solution of problems in the intermediate Knudsen number region.

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